Concepts and Examples
Introduction to Vectors – Part 2

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Find the dot product of two vectors.
2. Find unit vectors.
3. Express vectors in terms of the **i** and **j** unit vectors.
4. Find the angle between two vectors.
5. Determine if two vectors are parallel or perpendicular.

1. Definition of a Dot Product (1 of 5)

In this lesson, we discussed **scalar multiplication**. It results in a vector. The **dot product** is another type of vector multiplication. It results in a scalar which can be positive, negative, or 0.

Positive dot products tell us that that the angle between two vectors is less than 90^o therefore, the two vector point in a similar direction.

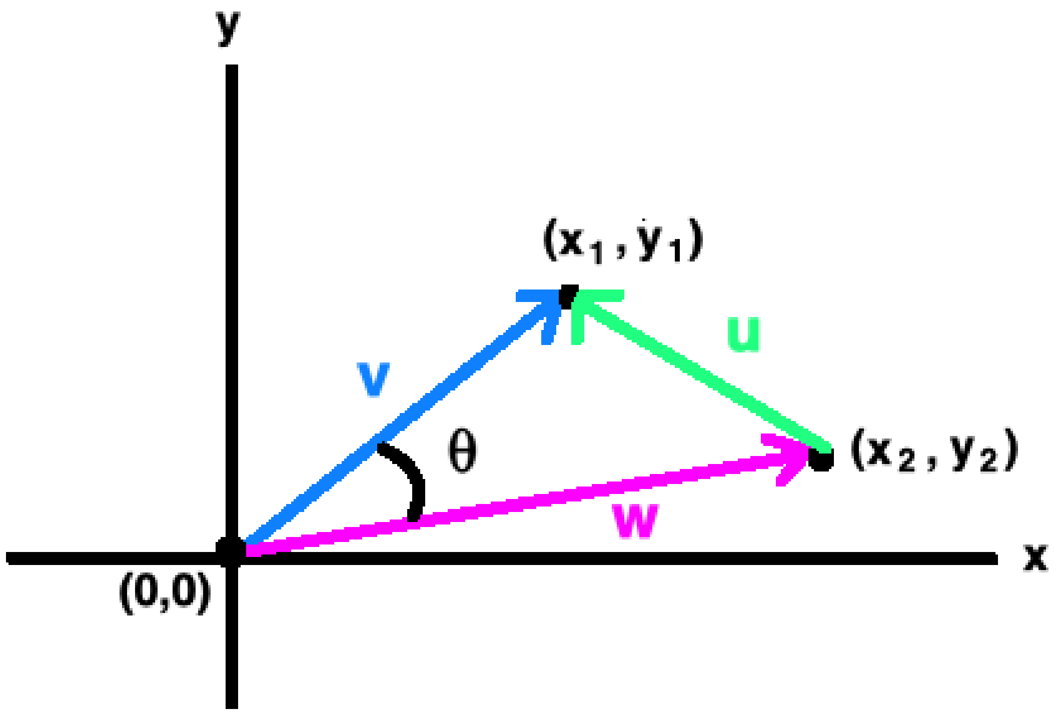
Negative dot products tell us that the angle between two vectors is greater than 90^o therefore, we can say that the two vectors roughly point in opposite directions.

A dot product of 0 tells us that the angle between two vectors is exactly 90^o, that is, the vectors are perpendicular.

We write the dot product of say, vector **v** and **w**, with a little dot between them, that is $\mathbf{v} \bullet \mathbf{w}$ and this is pronounced as “v dot w”.

Definition of a Dot Product (2 of 5)

The dot product is derived via the the *Law of Cosines*. You might want to review Lesson 15. We will first draw three vectors as follows:

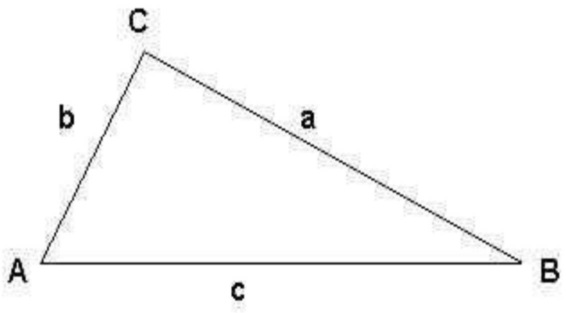


Please note that vector **v** and **w** are in standard position in a rectangular coordinate system. Their component form is $< x_1, y_1 >$ and $< x_2, y_2 >$, respectively. Since vector **u** is not in standard position, we must find its component form. You might want to review Lesson 18.

It is $< x_1 - x_2, y_1 - y_2 >$ ("terminal minus initial").

Definition of a Dot Product (3 of 4)

The *Law of Cosines* can be of the form $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$ where the angle $\boldsymbol{A}$ is opposite the side $\boldsymbol{a}$.



Looking at the picture on the previous slide, we will now use the angle $\boldsymbol{\theta}$ and the magnitude of vectors **u**, **v**, and **w** in the *Law of Cosine* formula. Specifically, we will let side ***a*** equal $\|\mathbf{u}\|$, side ***b*** equal $\|\mathbf{v}\|$, and side ***c*** equal $\|\mathbf{w}\|$ in the *Law of Cosine* formula.

$$\|\mathbf{u}\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad \|\mathbf{v}\| = \sqrt{x_1^{\,2} + y_1^{\,2}} \qquad \|\mathbf{w}\| = \sqrt{x_2^{\,2} + y_2^{\,2}}$$

Components of vectors **v** and **w** !!!

Definition of a Dot Product (4 of 5)

We end up with $||\mathbf{u}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$ ***cos*** $\boldsymbol{\theta}$ which can then be written as follows:

$$\left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)^2 = \left(\sqrt{x_1^{\,2} + y_1^{\,2}}\right)^2 + \left(\sqrt{x_2^{\,2} + y_2^{\,2}}\right)^2 \cos\theta$$

We are going to leave out the many steps that eventually lead us to the following equation:

$x_1 x_2 + y_1 y_2 = ||\mathbf{v}||\,||\mathbf{w}||$ ***cos*** $\boldsymbol{\theta}$ This must be memorized!

Both sides of the equal sign represents what we will call the **dot product** of the two vectors, in our case **v** and **w**! Please note that the left side of the equal sign consist of the components of the vectors **v** and **w**.

Definition of a Dot Product (5 of 5)

Example 1:

Given vector **a** = < 2, 5 > and vector **b** = < – 1, – 3 >, find the dot products **a** • **b** and **a** • **b.**

Since we are given vectors in component form, let's use the left side of the equation $x_1 x_2 + y_1 y_2 = ||\mathbf{v}||\ ||\mathbf{w}||\ \cos\theta$ to find the dot products.

We write **a** • **b** = 2(– 1) + 5(– 3) = – 2 + (– 15) = – 17.

Note that there is implied multiplication between the components.

We write **a** • **b** = (– 1)(2) + (– 3)(5) = – 2 + (– 15) = – 17.

Note that the result of a dot product is always the same no matter which vector is first and which is second.

2. The Angle between Two Vectors (1 of 3)

We just developed the equation $x_1 x_2 + y_1 y_2 = ||\mathbf{v}|| \, ||\mathbf{w}|| \, \cos \theta$ given vectors **v** and **w** are in standard position in a rectangular coordinate system. Their component form is $< x_1 , y_1 >$ and $< x_2 , y_2 >$, respectively.

We will now solve this equation for ***cos*** $\boldsymbol{\theta}$ to end up with the following:

$$\cos \theta = \frac{x_1 x_2 + y_1 y_2}{||\mathbf{v}|| \; ||\mathbf{w}||}$$

This must be memorized!

We can now use the arccosine to solve for angle $\boldsymbol{\theta}$, which is the angle between vectors **v** and **w**.

The Angle between Two Vectors (2 of 3)

Example 2:

Find the angle θ between the vectors **a** = < 4, 3 > and **b** = < 3, 5 >. Round to 1 decimal place.

We will use the formula $\cos\,\boldsymbol{\theta} = \dfrac{x_1 x_2 + y_1 y_2}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$ and find the following:

$\|\mathbf{a}\| = \sqrt{4^2 + 3^2} = 5$

$\|\mathbf{b}\| = \sqrt{3^2 + 5^2} = \sqrt{34}$ Do not round this number!

$a_1 a_2 + b_1 b_2 = 4(3) + 3(5) = 27$

The Angle between Two Vectors

Example 2 continued:

Then $\cos\theta = \dfrac{27}{5\sqrt{34}}$ and $\theta = \cos^{-1}\left(\dfrac{27}{5\sqrt{34}}\right) \approx 22.2^{o}$

NOTE: During the process of solving for the angle always try to use EXACT values in any formula! Do not round within a formula.

Specifically, in this problem use $\sqrt{34}$ in your calculator when you find the angle!

3. Unit Vectors (1 of 5)

A vector with a magnitude of 1 is called a **unit vector**. Following are two frequently used unit vectors.

1. **Standard Unit Vectors**

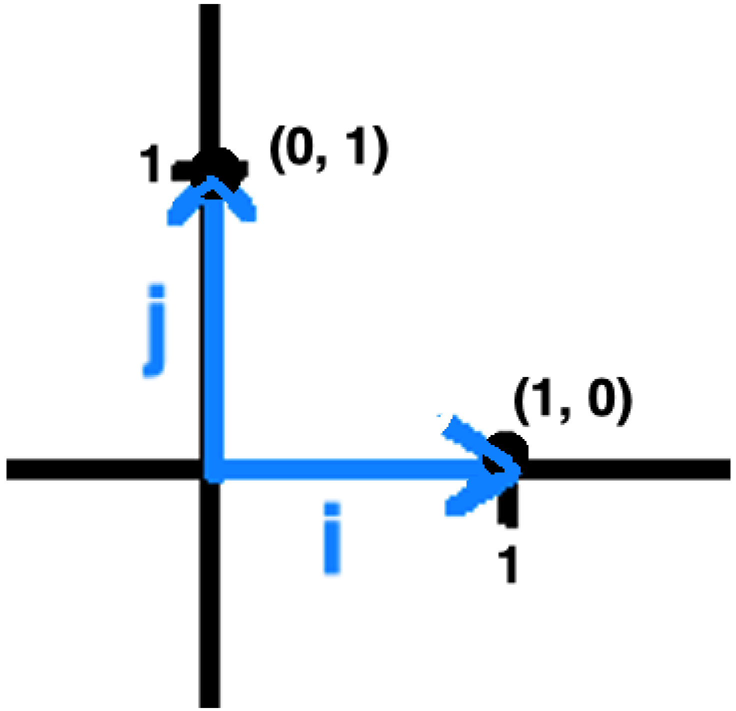
   Standard unit vectors have the special names **i** and **j** where **i** has component form < 1, 0 > and **j** has component form < 0, 1 >. This must be memorized!

   Earlier we were told that half arrows are placed over vector notation in handwritten documents.

   Given the standard unit vectors, we place a little "hat" over the **i** and **j** instead.
   That is, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Often we will pronounce this as "i hat" and "j hat”.

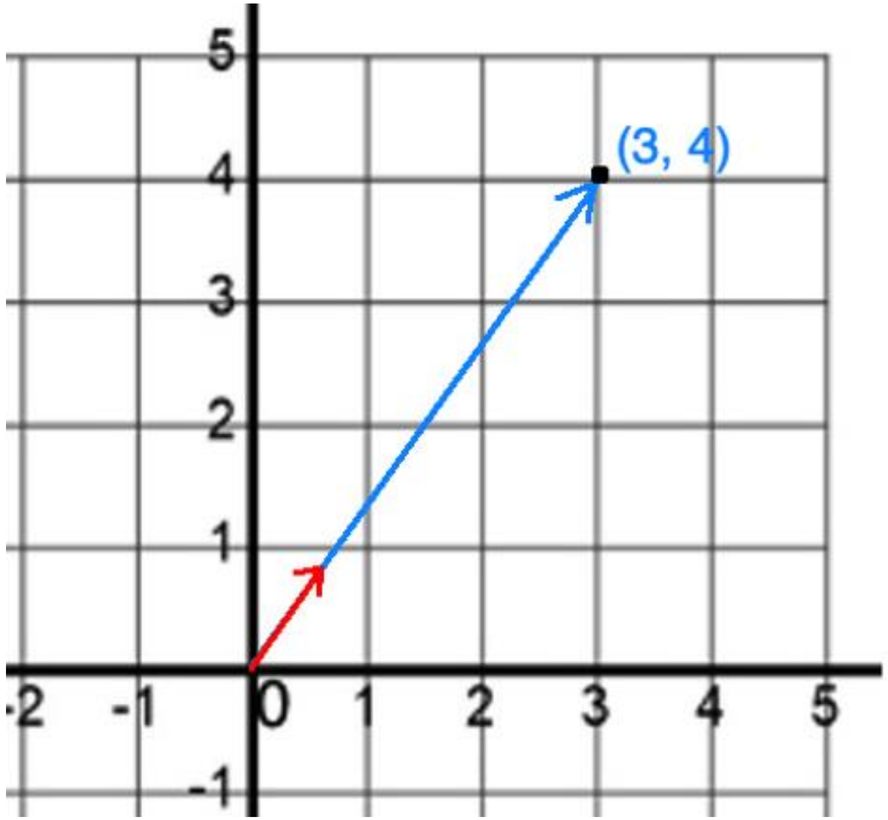
Unit Vectors (2 of 5)

Following is a picture of the unit vectors **i** and **j**.



Unit Vectors (3 of 5)

2. The other unit vectors we will discuss are **”unit vectors with the same direction as some given nonzero vector”**.



Let’s look at an example. The blue (long) vector **v** = < 3, 4 > has a **magnitude (length) of 5**.

A unit vector <u>in the same direction</u> of vector **v** must therefore lie on top of it. But it only has a **magnitude of 1**. This is illustrated by the red (short) vector. Please note that this vector is **one-fifth** as long as vector **v**.

Unit Vectors (4 of 5)

In our example, we scaled back the magnitude of **v** by 5, and we get a vector of magnitude 1. All we have to do now is divide the components of vector **v** by the same number. This gives us the components of the unit vector in the same direction as vector **v**.

So here is what we just did. Let's assume we are given a vector **v** with component form $< v_1 , v_2 >$. We find a unit vector with the same direction as vector **v** to be the following:

$$\left\langle \frac{v_1}{\|\mathbf{v}\|}, \frac{v_2}{\|\mathbf{v}\|} \right\rangle$$

This must be memorized!

The Angle between Two Vectors (5 of 5)

Example 3:

Find the component form of the unit vector in the same direction as vector **a** = < 4, 3 >.

We will find the magnitude of angle **a** and then divide both of its components by it.

$$\|\mathbf{a}\| = \sqrt{4^2 + 3^2} = 5$$

Next, we will divide the components 4 and 3 by 5. This will create the component form of the unit vector in the same direction as vector **a**.

$$\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

4. Express Vectors in Terms of the Standard Unit Vectors (1 of 4)

In many applications it is necessary to express vectors in component form in terms of standard unit vectors. This is done as follows:

Given vector **v** with component form < v_1 , v_2 >, we can also write vector **v** as v_1 **i** + v_2 **j**. Note, the components become coefficients of the unit vectors.

Please note that a vector written in component form does not include the **i** and **j** unit vectors and the components are within angle brackets and separated by commas.

However, a vector written in terms of the **i** and **j** unit vector is neither in angle brackets nor are the components separated by commas. Instead, we use their signs (positive, negative).

Express Vectors in Terms of the Standard Unit Vectors (2 of 4)

A little Proof:

v_1 **i** + v_2 **j** means $v_1 <1, 0> + v_2 <0, 1>$. Since v_1 and v_2 are scalars, we use scalar multiplication to get $< v_1 , 0> + <0, v_2 >$.

Then we use vector addition to get $< v_1 + 0, 0 + v_2 >$ which equals $< v_1 , v_2 >$.

Express Vectors in Terms of the Standard Unit Vectors (3 of 4)

Example 4:

Write the following vectors in terms of the standard unit vectors **i** and **j**:

(a) $\mathbf{v} = < 2, 5 >$

$\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ (no angle brackets using the signs of 2 and 5)

(b) $\mathbf{w} = < 5, -1 >$

$\mathbf{w} = 5\mathbf{i} - \mathbf{j}$ (no angle brackets using the signs of 5 and – 1)

Note, we do not write the coefficient – 1!

(c) $\mathbf{z} = < -6, -2 >$

$\mathbf{z} = -6\mathbf{i} - 2\mathbf{j}$ (no angle brackets using the signs of – 6 and – 2)

Express Vectors in Terms of the Standard Unit Vectors (4 of 4)

Example 5:

Write the following vectors in terms of the standard unit vectors **i** and **j**:

(a) **v** = < – 7, 0 >

v = – 7**i** (no angle brackets using the sign of – 7)

(b) **w** = < 0, 10 >

w = 10**j** (no angle brackets using the sign of 10)