Concepts and Examples
Sum and Difference Identities
Double and Half Angle Identities

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Given the formulas for the Sum Identities, be able to work with them.
2. Given the formulas for the Difference Identities, be able to work with them.
3. Given the formulas for the Double Angle Identities, be able to work with them.
4. Given the formulas for the Half Angle Identities, be able to work with them.

We are now going to discuss several identities, namely, the Sum and Difference identities and the Double and Half Angle Identities. Angles with names of ***u*** and ***v*** are used in these formulas. Please note that the variables ***u*** and ***v*** can be representatives of any algebraic expression !!!

These identities DO NOT have to be memorized! However, you must know how to work with them when given to you.

Proofs are provided separately for some of the formulas for your information only. Interested students might want to look at them. They can be found under the link "Point of Interest" in the Learning Materials in the MyOpenMath course.

1. Sum Identities for Sine and Cosine (1 of 2)

$$\sin(u+v) = \sin u \cos v + \sin v \cos u$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Sum Identities for Sine and Cosine (2 of 2)

Example 1:

Show that $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$ can be changed to $\tan\alpha + \tan\beta$.

Let's use the *Sum Identity for Sine* $\sin(u+v) = \sin u \cos v + \sin v \cos u$.
Here, $u \equiv \alpha$ and $v \equiv \beta$. Then we do the following:

$$\frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta} =$$

$$\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta} =$$ separated the fraction into two fractions

$$\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} =$$ canceled like factors

$$\tan\alpha + \tan\beta$$ used the *Quotient Identity*

2. Difference Identities for Sine and Cosine (1 of 2)

$$\sin(u - v) = \sin u \cos v - \sin v \cos u$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

Difference Identities for Sine and Cosine (2 of 2)

Example 2:

Show that $\cos\left(\frac{\pi}{2} - \theta\right)$ can be changed to ***sin*** θ.

Let's use the *Difference Identity for Cosine* $\cos(u - v) = \cos u \cos v + \sin u \sin v$.
Here, ***u*** $\equiv \frac{\pi}{2}$ and **v** $\equiv \theta$. Then we do the following:

$$\cos\left(\frac{\pi}{2}\right)\cos\theta + \sin\left(\frac{\pi}{2}\right)\sin\theta =$$

apply the *Difference Identity for Cosine*

$$0 \cdot \cos\theta + 1 \cdot \sin\theta =$$

evaluate cos $\left(\frac{\pi}{2}\right)$ and sin $\left(\frac{\pi}{2}\right)$

$$\sin\theta$$

3. Double Angle Identities for Sine and Cosine (1 of 3)

Double Angle Identity for Sine:

$$\sin 2u = 2\sin u \cos u$$

Double Angle Identities for Cosine:

Primary:

$$\cos 2u = \cos^2 u - \sin^2 u$$

Secondary:

- Replace $\cos^2 u$ in the primary with $1 - \sin^2 u$, we get $\cos 2u = 1 - 2\sin^2 u$.
- Replace $\sin^2 u$ in the primary with $1 - \cos^2 u$, we get $\cos 2u = 2\cos^2 u - 1$.

Double Angle Identities for Sine and Cosine

Example 3:

Simplify $\boldsymbol{6 \sin x \cos x}$ using the *Double Angle Identity for Sine* $\boldsymbol{\sin 2u = 2 \sin u \cos u}$.

Here, $\boldsymbol{u \equiv x}$. Let's factor out $\boldsymbol{3}$ to get $\boldsymbol{3(2 \sin x \cos x)}$, *which equals* $\boldsymbol{3(\sin 2x)}$.

Double Angle Identities for Sine and Cosine (3 of 3)

Example 4:

Simplify $\boldsymbol{4 - 8 \sin^2 x}$ using the secondary *Double Angle Identity for Cosine* $\boldsymbol{\cos 2u = 1 - 2 \sin^2 u}$.

Here, $\boldsymbol{u \equiv x}$. Let's factor out $\boldsymbol{4}$ to get $\boldsymbol{4(1 - 2 \sin^2 x)}$, *which equals* $\boldsymbol{4(\cos 2x)}$.

4. Half Angle Identities for Sine and Cosine (1 of 2)

Half Angle Identity for Sine:

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

Half Angle Identities for Cosine:

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

NOTE: The plus or minus sign in the half angle identities depends on the quadrant in which the angle $\frac{u}{2}$ lies.

Half Angle Identities for Sine and Cosine (2 of 3)

Example 5:

Rewrite $\boldsymbol{\sin 6x}$ in terms of a *Half Angle Identity*.

We will use the *Half Angle Identity for Sine* $\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$.

Here $\frac{\mathrm{u}}{2} \equiv \boldsymbol{6x}$, then $\boldsymbol{u \equiv 12x}$.

Using the pattern for the *Half Angle Identity for Sine*, we can state that $\boldsymbol{\sin 6x}$ equals $\pm\sqrt{\frac{1-\cos 12x}{2}}$.

Half Angle Identities for Sine and Cosine (3 of 3)

Example 6:

Simplify $\pm\sqrt{\frac{1+\cos 4x}{2}}$ using the *Half Angle Identity for Cosine* $\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$.

Here $\boldsymbol{u \equiv 4x}$, then $\frac{u}{2} \equiv \boldsymbol{2x}$.

Using the pattern for the *Half Angle Identity for Cosine*, we can state that

$\pm\sqrt{\frac{1+\cos 4x}{2}}$ equals $\boldsymbol{\cos 2x}$.