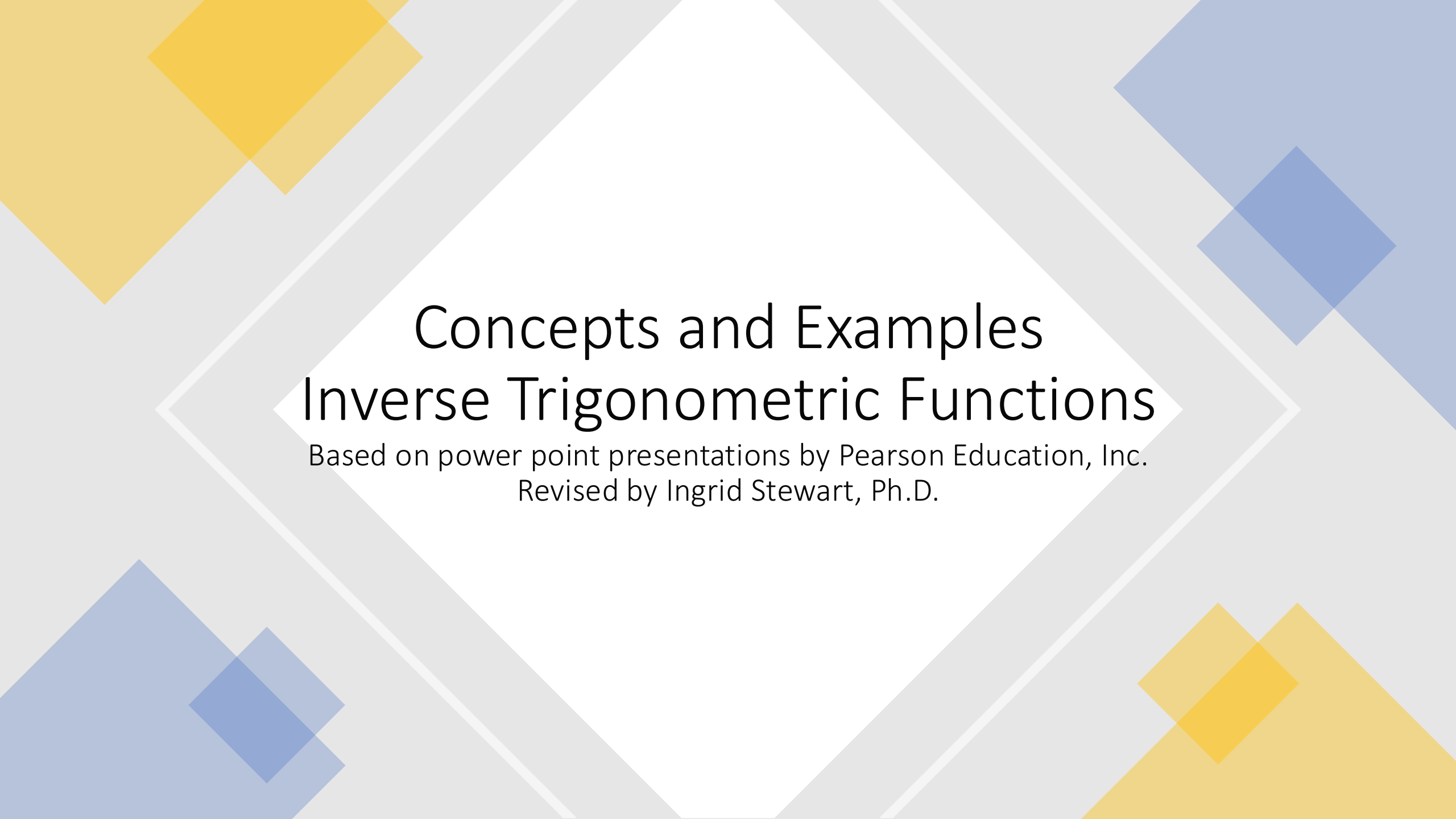




Concepts and Examples Inverse Trigonometric Functions

Based on power point presentations by Pearson Education, Inc.
Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Memorize the domain and range of the inverse sine function.
2. Memorize the domain and range of the inverse cosine function.
3. Memorize the domain and range of the inverse tangent function.
4. Use inverses to find angle measures.

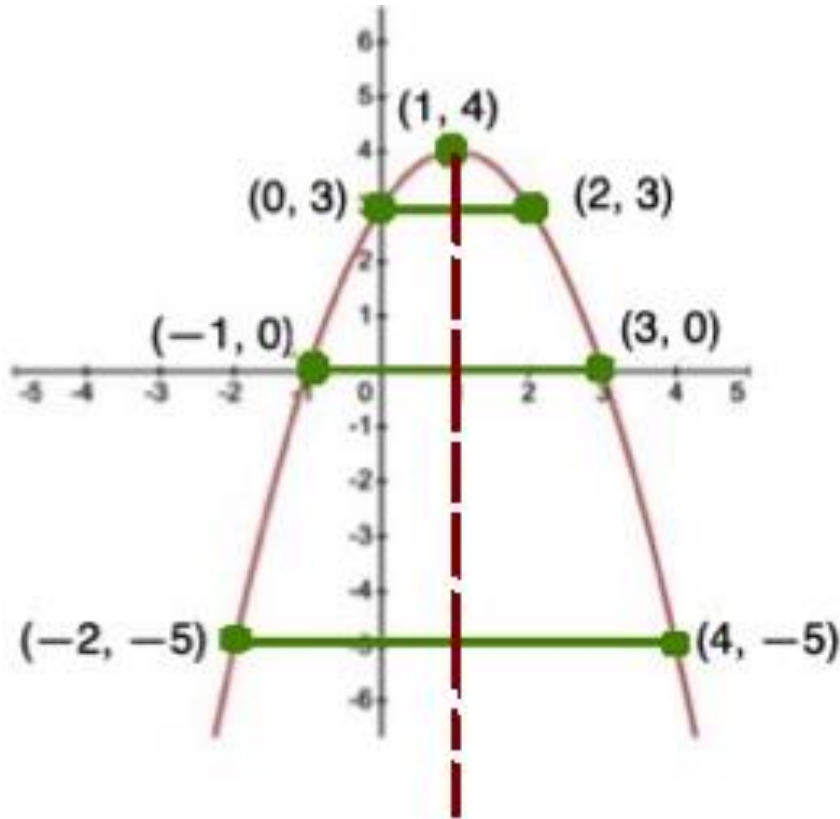
In this lesson, we will study inverse trigonometric functions. As a reminder, we will start out with a review of inverse functions in algebra.

From algebra you should remember that inverse functions are found by exchanging the dependent and independent variable. However, you can do this only if the function is **ONE-TO-ONE**.

One-to-one functions are special functions where the range values never repeat. We can determine that a function is NOT one-to-one if there is just one horizontal line crossing its graph intersecting more than once!

NOTE: During this exchange, the domain and range are also switched.

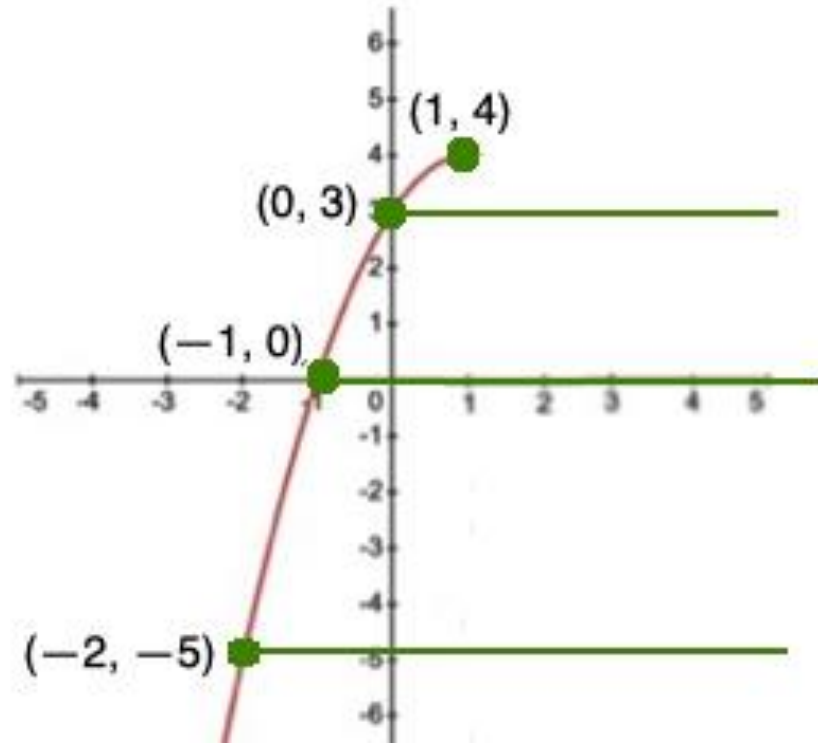
For instance, $p(x) = -(x - 1)^2 + 4$ is a quadratic function. Its DOMAIN consists of *all real numbers* and its RANGE consists of *numbers less than or equal to 1*. It is NOT one-to-one. Let's look at its graph.



Note the horizontal lines drawn into the graph. Their equations are $y = 3$, $y = 0$ and $y = -5$. As we can see, they pass through the points $(0, 3)$ and $(2, 3)$, $(-1, 0)$ and $(3, 0)$, and $(-2, -5)$ and $(4, -5)$.

This means, the range values 3, 0, and -5 repeat! As a matter of fact, all range values to the right and left of the dashed line (axis of symmetry) repeat

Now, let's we restrict the DOMAIN of $p(x) = -(x - 1)^2 + 4$, say to $x \leq 1$, then we have a function that is one-to-one. The RANGE still consists of *numbers less than or equal to 4*. Following is the graph.



NOTE: We can restrict the domain of a function that is NOT one-to-one in many different ways to make it become one-to-one. The example above is just one way of restricting the domain.

1. The Inverse of the Sine Function (1 of 5)

The sine function **$g(x) = \sin x$** is periodic and therefore NOT one-to-one. Just imagine drawing a horizontal line through its graph. This line will intersect with many points containing the same y -value.

To find its inverse, we must first restrict its domain which consists of all real numbers.

In trigonometry, we use the following DOMAIN restriction:

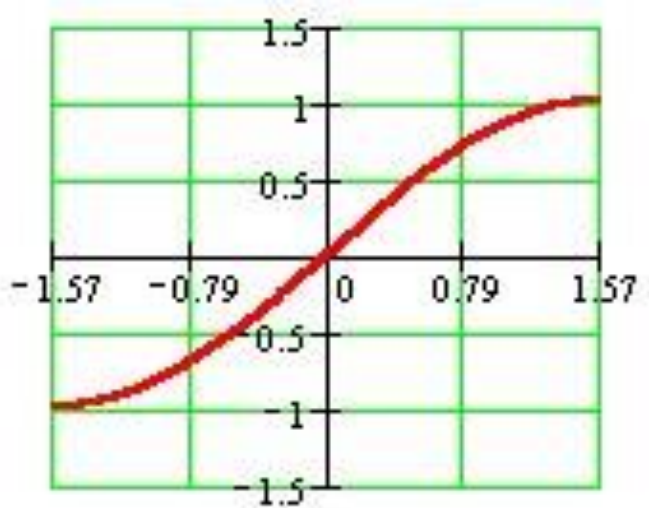
$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \text{ (radians)}$$

This gives us a RANGE of **$-1 \leq y \leq 1$** .

The Inverse of the Sine Function (2 of 5)

Below is the graph of the sine function with its domain restriction of

$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$



The function is now one-to-one, and we can find its inverse.

The Inverse of the Sine Function (3 of 5)

From algebra we know that the inverse of a function is found by exchanging the independent and dependent variables. Given **$g(x) = \sin x$** , we know that x is the independent variable and $g(x)$ is the dependent variable.

NOTE: In mathematics, we usually call the dependent variable y when given an independent variable of x .

That is, we will now let $g(x)$ equal y and write **$y = \sin x$** . Then we will exchange x and y to get **$x = \sin y$** . This is the inverse of the sine function.

The Inverse of the Sine Function (4 of 5)

However, in trigonometry we do not leave the inverse $x = \sin y$ in this form. It is rewritten as follows:

$$y = \arcsin x$$

$$\text{or } y = \sin^{-1} x$$

Both are pronounced either as *arcsine of x* or as *sine inverse of x*.

We know from algebra, that the domain of a given function is the range of its inverse function, and the range of a given function is the domain of its inverse function. Inverses of functions must also be one-to-one functions.

The Inverse of the Sine Function (5 of 5)

Therefore, we can say the following about the domain and range of the inverse sine function:

Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (angles can be radians but also in degrees!)

Please note, that the range values consist of negative angles in QIV and positive angles in QI !!!

YOU MUST MEMORIZE THE DOMAINS AND RANGES OF THE INVERSE TRIGONOMETRIC SINE FUNCTION. IT WILL BECOME IMPORTANT VERY SOON.

2. The Inverse of the Cosine Function (1 of 4)

The cosine function $h(x) = \cos x$ is periodic and therefore NOT one-to-one. Just imagine drawing a horizontal line through its graph. This line will intersect with many points containing the same y -value.

To find its inverse, we must first restrict its domain which consists of all real numbers.

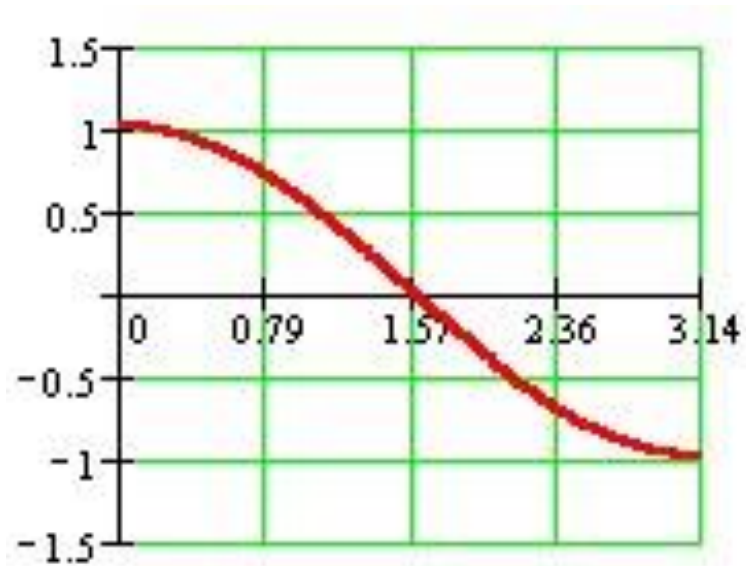
In trigonometry, we use the following DOMAIN restriction:

$$0 \leq x \leq \pi \text{ (radians)}$$

This gives us a RANGE of $-1 \leq y \leq 1$.

The Inverse of the Cosine Function (2 of 4)

Below is the graph of the cosine function with its domain restriction of $0 \leq x \leq \pi$.



The function is now one-to-one, and we can find its inverse.

The Inverse of the Cosine Function (3 of 4)

We will now let $h(x)$ equal y and write $y = \cos x$. Then we will exchange x and y to get $x = \cos y$. This is the inverse of the cosine function.

However, in trigonometry we do not leave the inverse $x = \cos y$ in this form. It is rewritten as follows:

$$y = \arccos x$$

$$\text{or } y = \cos^{-1} x$$

Both are pronounced either as *arccosine of x* or as *cosine inverse of x* .

The Inverse of the Cosine Function (4 of 4)

The domain and range of the inverse cosine function are as follows:

Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$ (angles can be radians but also in degrees!)

Please note, that the range values consist of positive angles in QI and in QII !!!

YOU MUST MEMORIZE THE DOMAINS AND RANGES OF THE INVERSE TRIGONOMETRIC COSINE FUNCTION. IT WILL BECOME IMPORTANT VERY SOON.

3. The Inverse of the Tangent Function (1 of 4)

The tangent function **$f(x) = \tan x$** is periodic and therefore NOT one-to-one. Just imagine drawing a horizontal line through its graph. This line will intersect with many points containing the same y-value.

To find its inverse, we must first restrict its domain which consists of all real numbers.

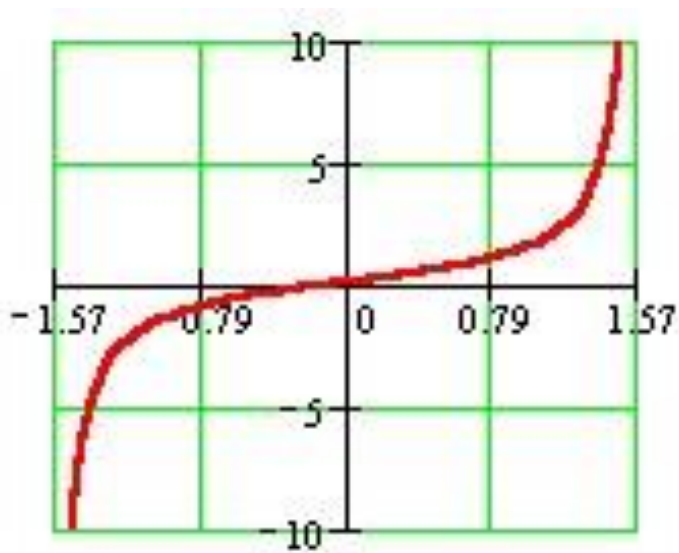
In trigonometry, we use the following DOMAIN restriction:

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \text{ (radians)}$$

This gives us a RANGE consisting of *All Real Numbers*.

The Inverse of the Tangent Function (2 of 4)

Below is the graph of the tangent function with its domain restriction of $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



The function is now one-to-one, and we can find its inverse.

The Inverse of the Tangent Function (3 of 4)

We will now let $f(x)$ equal y and write $y = \tan x$. Then we will exchange x and y to get $x = \tan y$. This is the inverse of the tangent function.

However, in trigonometry we do not leave the inverse $x = \tan y$ in this form. It is rewritten as follows:

$$y = \arctan x$$

$$\text{or } y = \tan^{-1} x$$

Both are pronounced either as *arctangent of x* or as *tangent inverse of x* .

The Inverse of the Tangent Function (4 of 4)

The domain and range of the inverse tangent function are as follows:

Domain: *All Real Numbers*

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (angles can be radians but also in degrees!)

Please note, that the range values consist of negative angles in QIV and positive angles in QI !!!

YOU MUST MEMORIZE THE DOMAINS AND RANGES OF THE INVERSE TRIGONOMETRIC TANGENT FUNCTION. IT WILL BECOME IMPORTANT VERY SOON.

4. The Inverses of the Cotangent, Secant, and Cosecant Functions

For completeness's sake, the definitions of the inverse cotangent, secant, and cosecant functions will be shown. However, we are not going to work with them in this course.

$$y = \text{arccot } x \text{ or } y = \cot^{-1} x$$

$$\text{Domain: } -\infty < x < \infty$$

$$\text{Range: } 0 < y < \pi$$

$$y = \text{arcsec } x \text{ or } y = \sec^{-1} x$$

$$\text{Domain: } |x| \geq 1$$

$$\text{Range: } 0 < y < \pi$$

$$y = \text{arccsc } x \text{ or } y = \csc^{-1} x$$

$$\text{Domain: } |x| \geq 1$$

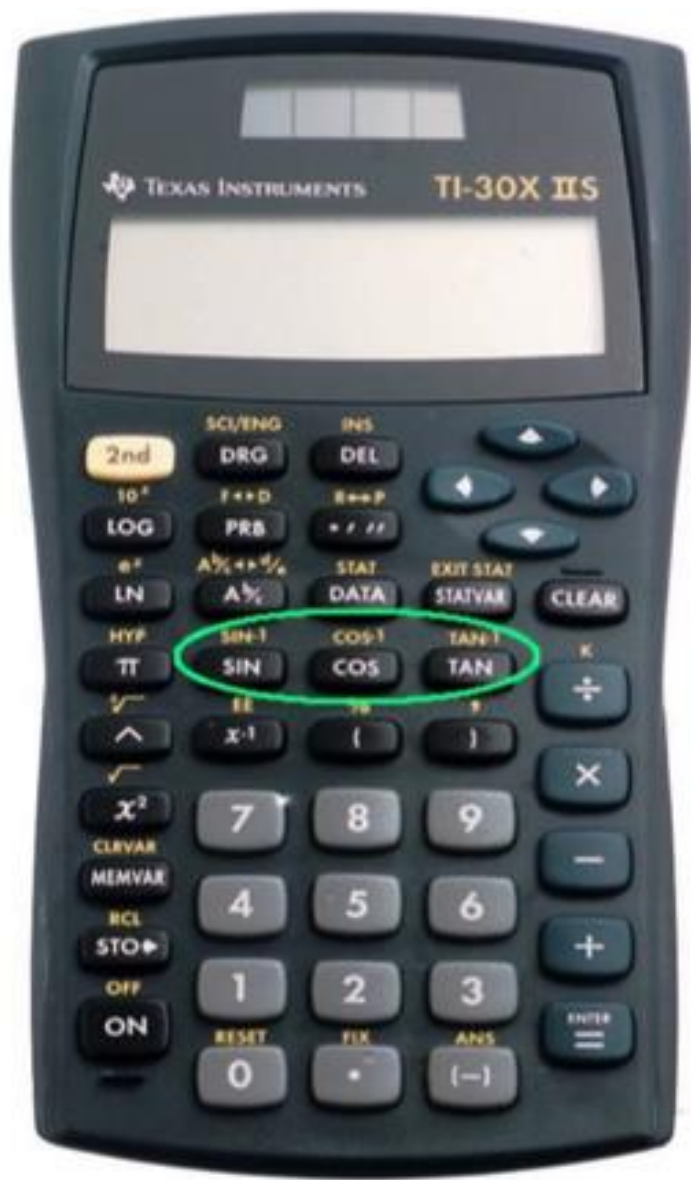
$$\text{Range: } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

5. Using Inverses to Find Angle Measures (1 of 9)

Now that we know about inverse trigonometric functions, we can find angles given a trigonometric ratio. For this we must often use a calculator. Let's look at the TI-30X IIS scientific calculator.

It is important to note that the calculator is programmed to only give angles lying in the range of the arcsine, arccosine, and arctangent function.

Using Inverses to Find Angle Measures (2 of 9)



The inverse trigonometric functions are above the sin, cos, and tan buttons! See picture!

We activate the $\sin^{-1}$ (arcsine), $\cos^{-1}$ (arccosine), or $\tan^{-1}$ (arctangent) function by pressing the 2nd button followed by the “sin”, “cos”, or “tan” butt

Left parenthesis will open when you activate the arcsine, arccosine, or arctangent. Specifically, you will see $\sin^{-1}($, $\cos^{-1}($, or $\tan^{-1}($.

After you type the numeric value, you MUST type the right parenthesis, namely $)$, before you press ENTER.

Be sure that the calculator is in the correct mode!

Using Inverses to Find Angle Measures (3 of 9)

For the arcsine, the angles produced by the calculator will lie in the following range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or $-90^\circ \leq y \leq 90^\circ$ (positive angles in QI and negative angles in QIV)

For the arccosine, the angles produced by the calculator will lie in the following range: $0 \leq y \leq \pi$ or $0^\circ \leq y \leq 180^\circ$ (positive angles in QI and positive angles in QII)

For the arctangent, the angles produced by the calculator will lie in the following range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$ or $-90^\circ < y < 90^\circ$ (positive angles in QI and negative angles in QIV)

Using Inverses to Find Angle Measures (4 of 9)

Example 1:

Given $y = \sin x$, let $x = 30^\circ$, then find $\sin 30^\circ$ using the calculator.

The calculator must be in degree mode. We find that $\sin 30^\circ = 0.5$.

Given $y = \arcsin x$, let $x = 0.5$, then find $\arcsin (0.5)$ using the calculator.

We find that $\arcsin 0.5 = 30^\circ$.

Please note that $\arcsin (0.5)$ gives us back an angle of 30° ! This is because 30° is an angle in QI.

Using Inverses to Find Angle Measures (5 of 9)

Example 2:

Given $y = \cos x$, let $x = 60^\circ$, then find $\cos 60^\circ$ using the calculator.

The calculator must be in degree mode. We find that $\cos 60^\circ = 0.5$.

Given $y = \arccos x$, let $x = 0.5$, then find $\arccos (0.5)$ using the calculator.

We find that $\arccos (0.5) = 60^\circ$.

Please note that $\arccos (0.5)$ gives us back an angle of 60° ! This is because 60° is an angle in QI.

Using Inverses to Find Angle Measures (6 of 9)

Example 3:

Given $y = \tan x$, let $x = 45^\circ$, then find $\tan 45^\circ$ using the calculator.

The calculator must be in degree mode. We find that $\tan 45^\circ = 1$.

Given $y = \arctan x$, let $x = 1$, then find $\arctan (1)$ using the calculator.

We find that $\arctan (1) = 45^\circ$.

Please note that $\arctan (1)$ gives us back an angle of 45° ! This is because 45° is an angle in QI.

Using Inverses to Find Angle Measures (7 of 9)

Example 4:

Given $y = \sin x$, let $x = 210^\circ$, then find $\sin 210^\circ$ using the calculator.

The calculator must be in degree mode. We find that $\sin 210^\circ = -0.5$.

Given $y = \arcsin x$, let $x = -0.5$, then find $\arcsin(-0.5)$ using the calculator.

We find that $\arcsin(-0.5) = -30^\circ$.

Please note that $\arcsin(-0.5)$ DOES NOT give us back an angle of 210° !

This is because 210° is an angle in QIII, and the calculator can only give us angles in QI and QIV for arcsine. Angle -30° does lie in QIV!

For later work it is important to note that both 210° and -30° have a reference angle of 30° .

Using Inverses to Find Angle Measures (8 of 9)

Example 5:

Given $y = \cos x$, let $x = 240^\circ$, then find $\cos 240^\circ$ using the calculator.

The calculator must be in degree mode. We find that $\cos 240^\circ = -0.5$.

Given $y = \arccos x$, let $x = -0.5$, then find $\arccos(-0.5)$ using the calculator.

We find that $\arccos(-0.5) = 120^\circ$.

Please note that $\arccos(-0.5)$ DOES NOT give us back an angle of 240° !

This is because 240° is an angle in QIII, and the calculator can only give us angles in QI and QII for arccosine. Angle 120° does lie in QII!

For later work it is important to note that both 240° and 120° have a reference angle of 60° .

Using Inverses to Find Angle Measures (9 of 9)

Example 6:

Given $y = \tan x$, let $x = 135^\circ$, then find $\tan 135^\circ$ using the calculator.

The calculator must be in degree mode. We find that $\tan 135^\circ = -1$.

Given $y = \arctan x$, let $x = -1$, then find $\arctan(-1)$ using the calculator.

We find that $\arctan(-1) = -45^\circ$.

Please note that $\arctan(-1)$ DOES NOT give us back an angle of 135° ! This is because 135° is an angle in QII, and the calculator can only give us angles in QI and QIV for arctangent. Angle -45° does lie in QIV!

For later work it is important to note that both 135° and -45° have a reference angle of 45° .