Concepts and Examples
Introduction to Angles – Part 1

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

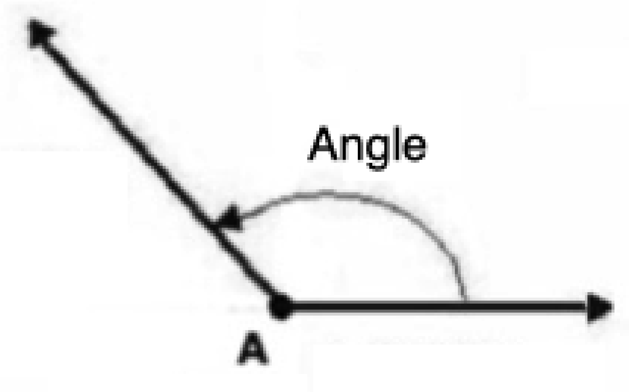
Learning Objectives

1. Define angles.
2. Measure angles in degrees.
3. Measure angles in radians.
4. Know the difference between positive and negative angles.

1. Definition of an Angle (1 of 4)

An angle is determined by rotating a ray a specific distance about its starting point. The following is a ray. It has a **starting point** (in the picture below it is called ***A***) and then it extends forever in one direction, say to the right.

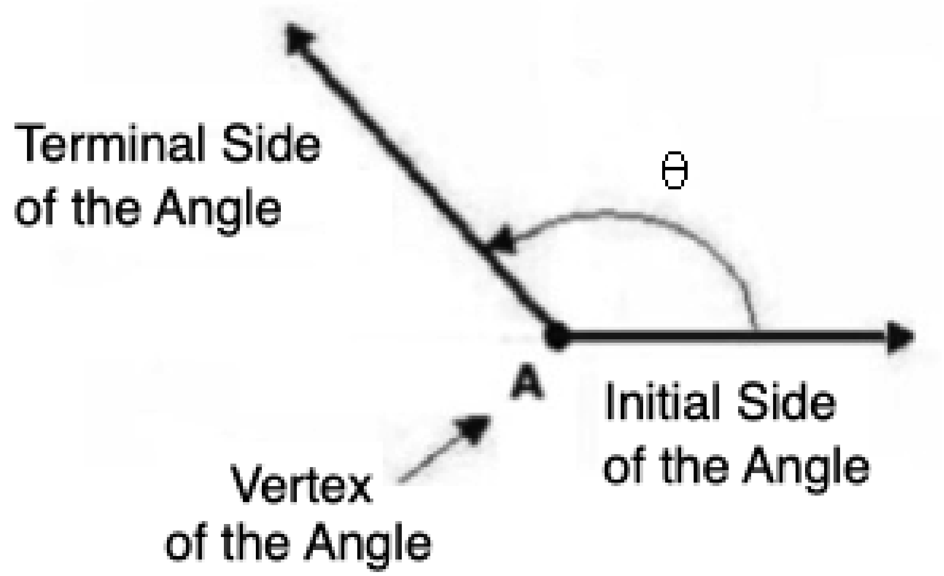
Imagine we rotate this ray around its starting point. We will end up with an angle as follows:



NOTE: When drawing an angle, the rotation of the ray about its starting point is usually indicated with an arc in between the initial and terminal sides.

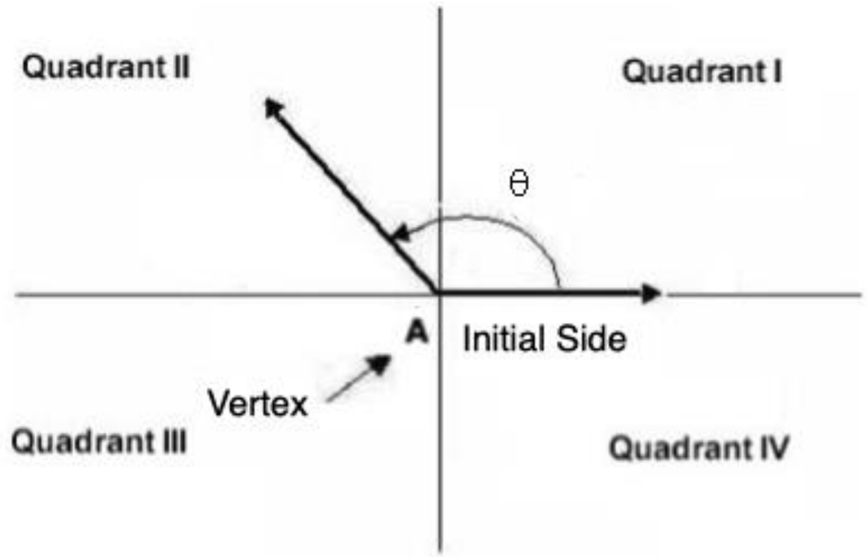
Definition of an Angle (2 of 4)

Notice in the picture below, that the original ray is now called the **initial side** of the angle. The ray we created by the rotation is called the **terminal side** of the angle. The point where the initial and the terminal side meet is called the **vertex** of the angle. Note that we gave the angle a name. We often use the Greek letter **theta** "θ" as an angle name.



Definition of an Angle (3 of 4)

In trigonometry, we usually place an angle into a *Rectangular Coordinate System* with its Vertex at the Origin (0, 0), and its initial side along the positive horizontal axis. We then say that the angle is in **standard position**.



Definition of an Angle

Common Names of Angles

Often Greek letters are used to represent unknown angles. We already encountered the angle name theta θ. Other common names are

Alpha: α Beta: β Gamma: γ

Sometimes, angles are given the name of their vertex. In the previous angle pictures, the angle vertex was called **A**, therefore, we could name the angle **A** as well.

2. Angle Measure in Degrees (1 of 6)

The measure of an angle is often called **magnitude**. In trigonometry, we measure angles in degrees or radians. We'll discuss degrees first.

We know that an angle is measured in degrees when there is little circle in the upper right-hand corner of a number. For example, ***45º*** which is then pronounced "*45 degrees*".

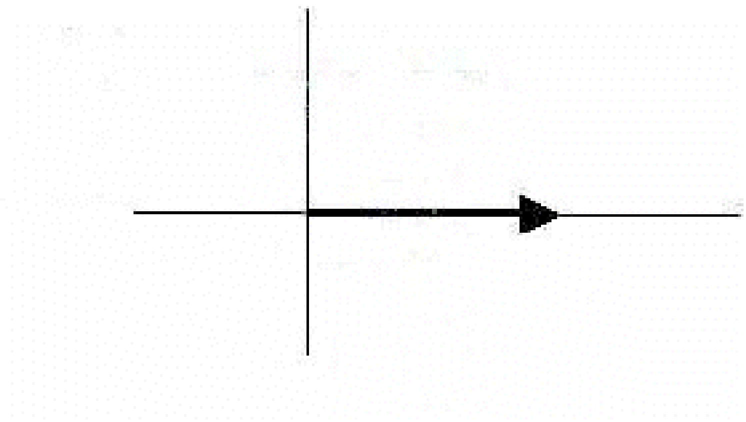
In trigonometry, angle magnitudes can range from 0^o to infinity. For example, it is not unusual to deal with a $9{,}850^o$ angle.

Degrees can further be divided into minutes (**'**) and seconds (**"**).

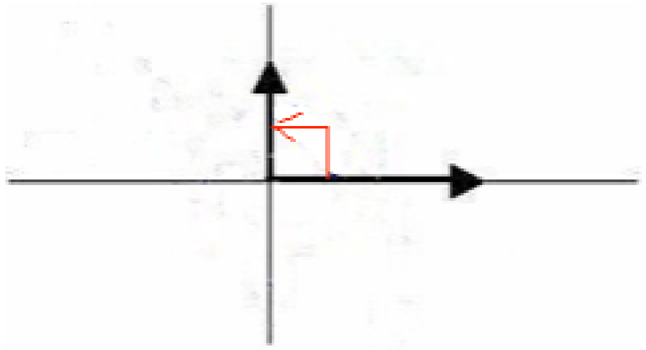
> ***1º = 60'*** (minutes) **using the apostrophe on the computer keyboard**
>
> ***1' = 60"*** (seconds) **using the quotation mark on the keyboard**

Angle Measure in Degrees (2 of 6)

There are several angles that we use often in trigonometry. There is the **Zero-Degree Angle**, the **Right Angle**, the **Straight Angle**, the **Full Angle,** the **Reflex Angle**, the **Acute Angle**, and the **Obtuse Angle**. Let's look at some of their pictures.

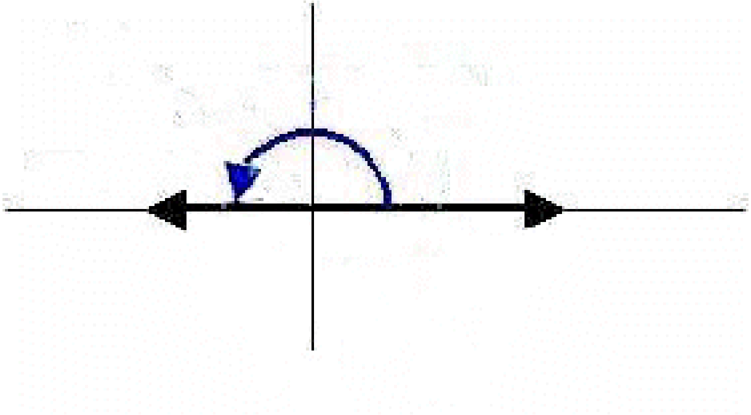


Zero-Degree Angle
Notice that the initial side and the terminal side are the same. There is NO arc representing an angle.

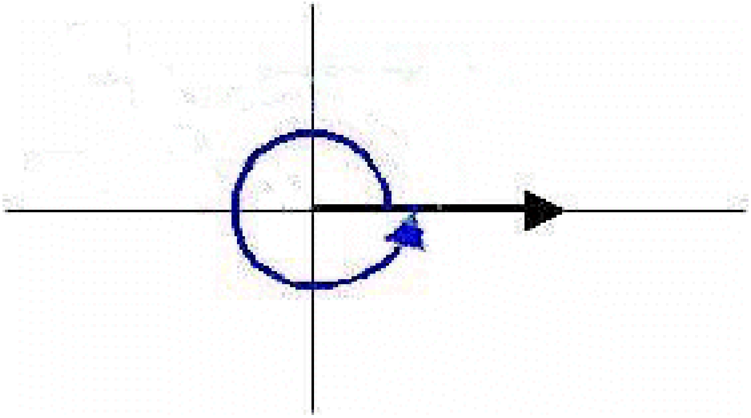


Right Angle
Its measure is 90^{o}. Notice that instead of an arc we often use a square to represent the angle.

Angle Measure in Degrees (3 of 6)

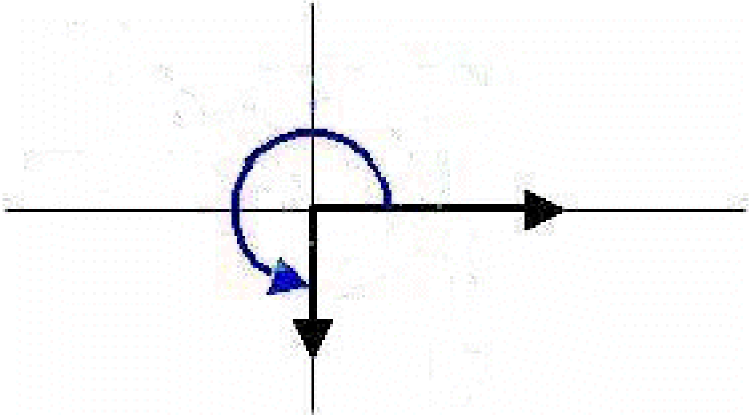


Straight Angle
Its measure is 180^o.



Full Angle
Its measure is 360^o. Notice that the initial side and the terminal side are the same just like in the Zero-Degree Angle. However, there is now an arc representing the angle.

Angle Measure in Degrees (4 of 6)



Reflex Angle

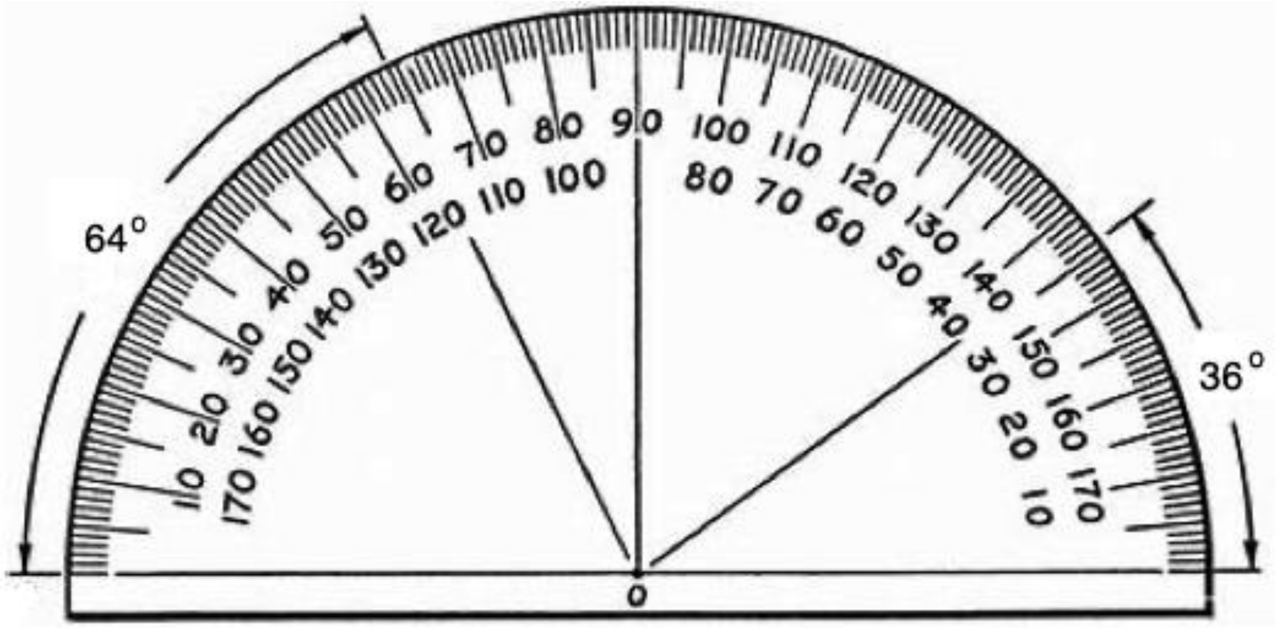
Its measure is greater than 180^o but less than 360^o. The measure in the angle on the left is 270^o.

Angle Measure in Degrees (5 of 6)

Acute Angles

Angles whose measure is greater than 0^o but less than 90^o.

Below is a picture of a protractor. It shows angles 36^o and 64^o. Note specifically the horizontal **Base**, the **Center 0**, and the **Scale**. When measuring angles, the **Center** is placed at the vertex of the angle with the **Base** along its initial side.



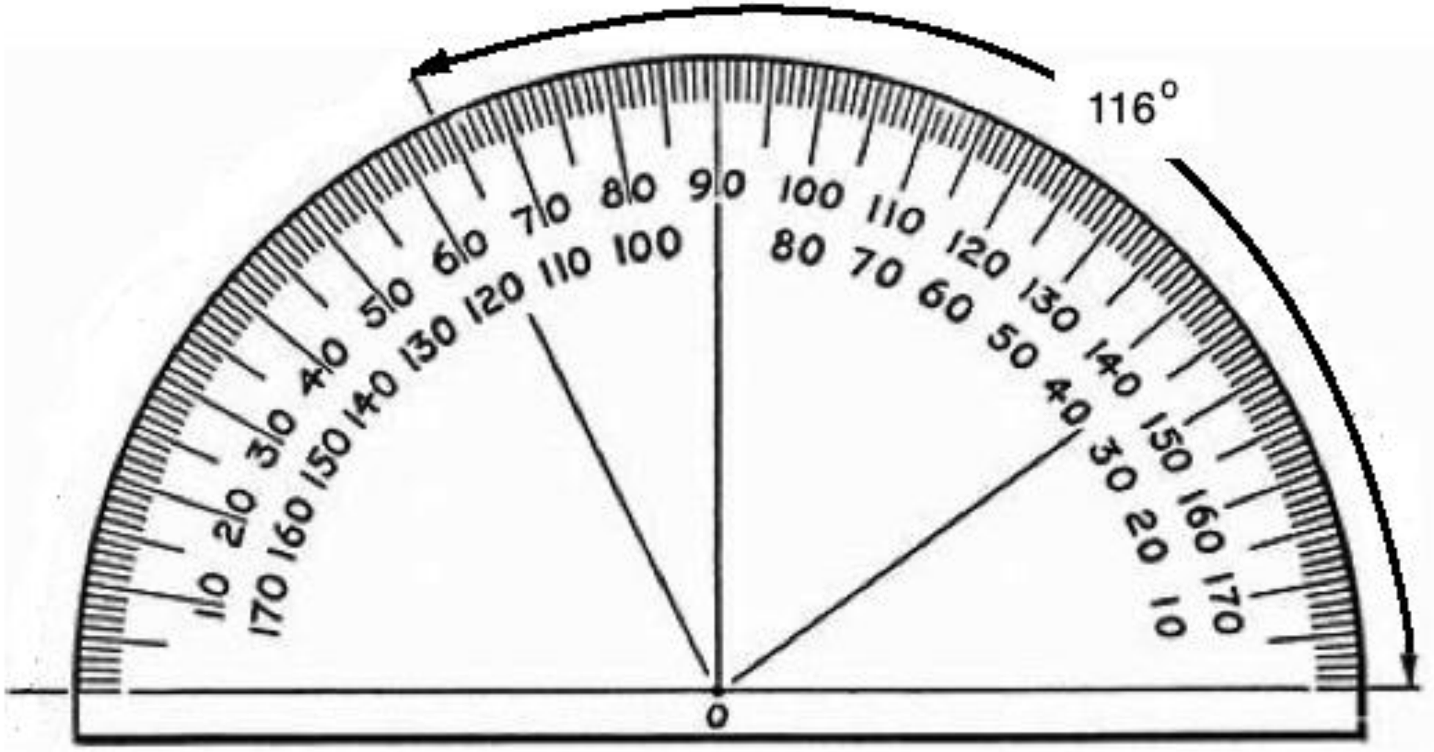
Note that there are two scales. The outer one is for measuring acute angles. They can be measured from the right or left of the Center.

Angle Measure in Degrees (6 of 6)

Obtuse Angles

Angles whose measure is greater than 90^o but less than 180^o.

Below is a picture of a protractor. It shows angle 116^o.



Note there is a second scale for measuring obtuse angles. It is the inside one. We can only measure obtuse angle from the right of Center.

3. Angle Measure in Radians (1 of 10)

Unlike degrees, which use the "circle" symbol, radian measure use NO symbol. Later, in context, you will know when you are working with radians.

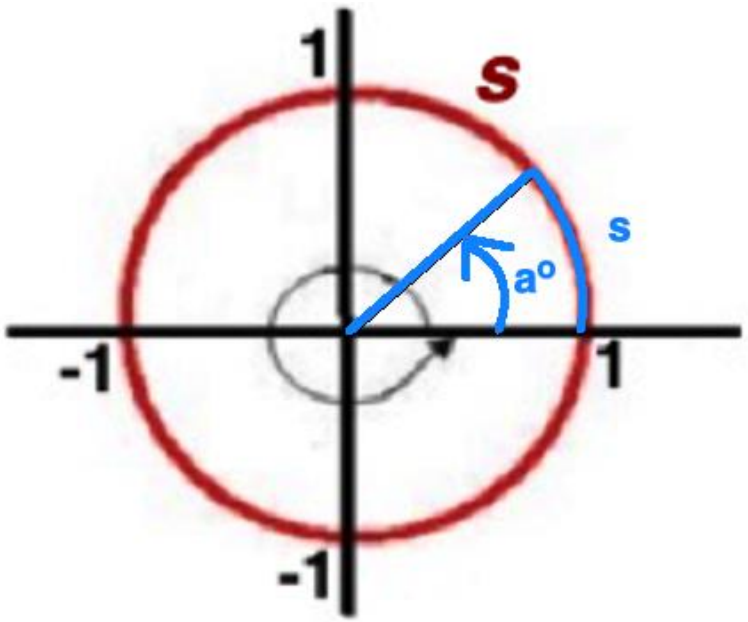
Radians are actually connected to the circumference (perimeter) of a circle with a radius ***r*** equal to ***1***.

> NOTE: We know from geometry that the circumference formula of a circle is $\mathbf{2\pi}\boldsymbol{r}$ where $\boldsymbol{\pi}$ (pi) is the famous number approximately equal to 3.14 and ***r*** is the radius of the circle.

Given a circle with $\boldsymbol{r} = \mathbf{1}$, the circumference of this circle is $\mathbf{2\pi(1)}$ or simply $\mathbf{2\pi}$.

Angle Measure in Radians (2 of 10)

Let's draw a circle with **radius 1** into a *Rectangular Coordinate System* with center at the origin. Next, let's place an acute angle with some general measure of ***a* degrees (*a*°)** at its center as follows:

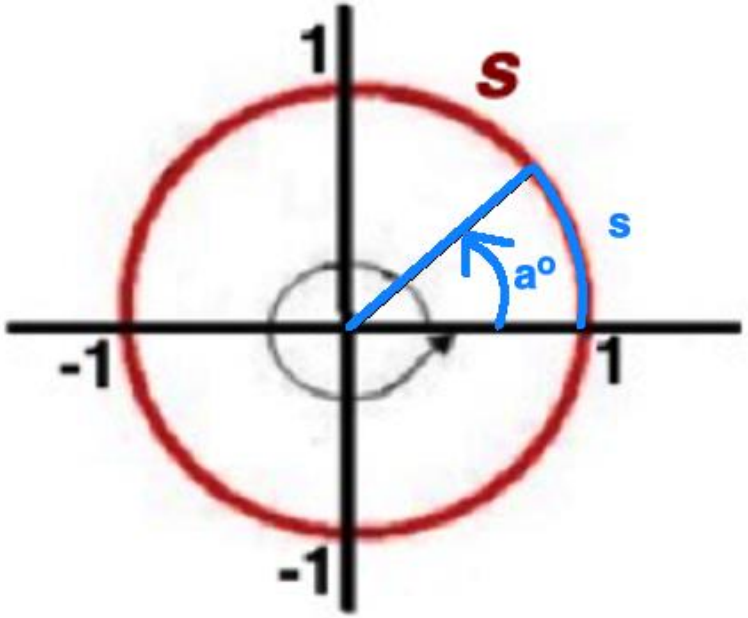


Note that the angle ***a*°** is opposite an arc ***s*** on the circle circumference.

The length of the arc ***s*** is considered to be the radian equivalent of the angle with measure ***a*°**.

Angle Measure in Radians

To get us started with degree to radian equivalents, let's draw another circle with **radius 1** into a *Rectangular Coordinate System*, but now let's let's look at the angle with measure **360^o**.



It should then be obvious that the size of the arc ***s*** opposite this angle is **2π** (see earlier discussion).

2π is then considered to be the radian equivalent of the angle with measure **360^o**.

Angle Measure in Radians

We just found that 360^o is equivalent to 2π (radians). This is written as $360^o \equiv 2\pi$. This must be memorized. Incidentally, we use the "equivalent" sign $\equiv$ instead of the "equal" sign because we are actually comparing angles to arc lengths.

Note that 2π is approximately equal to $2(3.14) = 6.28$. However, we usually DO NOT convert radians containing a factor of π into decimal form.

Two important equivalencies which must be memorized can be derived arithmetically from $360^o \equiv 2\pi$. Specifically,

$$1^\circ \equiv \frac{\pi}{180} \quad \text{(radian)}$$

and

$$\left(\frac{180}{\pi}\right)^\circ \equiv 1 \quad \text{(radian)}$$

Angle Measure in Radians

Example 1:

Express ***164º*** in EXACT radians (π is included in the answer!). Reduced to lowest terms, if necessary. Do not use the calculator!

We will use the conversion $1^\circ \equiv \dfrac{\pi}{180}$, and we simply multiply both sides by ***164º***.

$$164^\circ \equiv 164\left(\frac{\pi}{180}\right) = \frac{164\pi}{180}$$

We notice that ***4*** divides both into the numerator and the denominator. Therefore, we can reduce the answer as follows:

$$164^\circ \equiv \frac{41\pi}{45}$$

Angle Measure in Radians (6 of 10)

Example 2:

Express $\frac{11\pi}{36}$ in EXACT degrees. Reduced to lowest terms, if necessary. Do not use the calculator!

Since we do not see a degree symbol, we must assume that we are dealing with radians.

Therefore, we will use the conversion $\left(\frac{180}{\pi}\right)^{\circ} \equiv 1$. We will multiply both sides by $\frac{11\pi}{36}$.

$$\frac{11\pi}{36}\left(\frac{180}{\pi}\right)^{\circ} \equiv \frac{11\pi}{36}$$

Angle Measure in Radians

Example 2 continued:

Next, we reduce the fraction on the left of the equal sign to get the following:

$$\frac{11\cancel{\pi}}{{}_{1}\cancel{36}}\left(\frac{\overset{5}{\cancel{180}}}{\cancel{\pi}}\right)^{\circ} \equiv \frac{11\pi}{36}$$

We find that $55^{\circ} \equiv \dfrac{11\pi}{36}$.

Angle Measure in Radians (8 of 10)

Why use Radians?

As you move through more advanced topics in mathematics, physics, and engineering, you will discover that there are certain relationships that only work when radians are used.

Is Radian Measure always Expressed in terms of Pi?

In trigonometry, we like to work with radian measures containing a factor of π, for example 2π or $\frac{\pi}{180}$, etc. We call them EXACT values.

However, be aware that in your studies you will also see radian measures expressed without a factor of π. They would look just like regular numbers, for example 4, 1.5, $\frac{1}{2}$, etc.

Angle Measure in Radians (9 of 10)

Example 3:

Express $\frac{1}{3}$ in EXACT degrees. Reduced to lowest terms, if necessary. Do not use the calculator!

Since we do not see a degree symbol, we must assume that we are dealing with radians.

Therefore, we will use the conversion $\left(\frac{180}{\pi}\right)^{\circ} \equiv 1$. We multiply both sides by $\frac{1}{3}$ to get the following:

$$\frac{1}{3}\left(\frac{180}{\pi}\right)^{\circ} \equiv \frac{1}{3}$$

Angle Measure in Radians

Example 3 continued:

Next, we reduce the fraction on the left of the equal sign to get

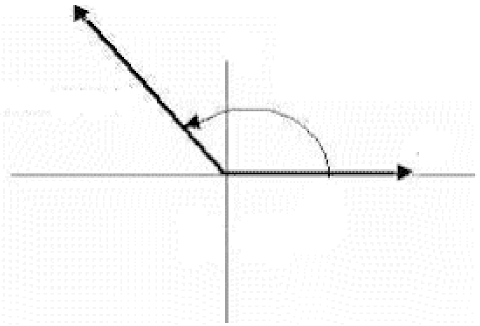
$$\left(\frac{60}{\pi}\right)^{\circ} = \frac{1}{3}$$

Just like you will see radian measures expressed without a factor of π., you will also see degrees expressed in terms of radians.

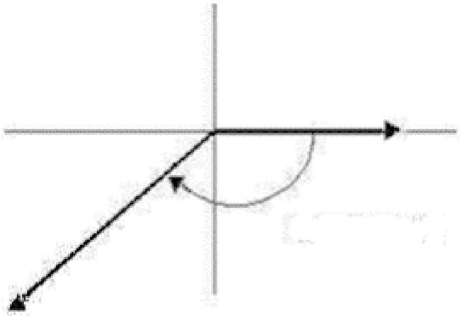
We say that $\left(\frac{60}{\pi}\right)^{\circ}$ is an exact degree measure.

4. Positive and Negative Angles (1 of 5)

In trigonometry, angles can have a positive or negative measure. **Angles with positive measure** are indicated by an arc drawn in counter-clockwise rotation. Please note the arrow at the end of the arc!



Angles with negative measure are indicated by an arc drawn in clockwise rotation. Please note the arrow at the end of the arc!



Positive and Negative Angles (2 of 5)

A word about negative angles ...

Please note that as a purely numerical statement – 20 is less than 10. However, in case of angles, the minus sign simply indicates orientation. Therefore, we may state that an angle measuring 10^o has a smaller magnitude than an angle measuring -20^o.

Positive and Negative Angles (3 of 5)

Example 4:

Express ***– 540º*** in EXACT radians (π is included in the answer!). Reduced to lowest terms, if necessary. Do not use the calculator!

We will use the conversion $1^\circ \equiv \frac{\pi}{180}$, and we simply multiply both sides by ***– 540º*** .

$$-540^\circ \equiv -540\left(\frac{\pi}{180}\right) = -\frac{540\pi}{180}$$

We notice that ***180*** divides both into the numerator and the denominator. Therefore, we can reduce the answer as follows:

$$-540^\circ \equiv 3\pi$$

Angle Measure in Radians

Example 5:

Express $-\frac{4\pi}{3}$ in EXACT degrees. Reduced to lowest terms, if necessary. Do not use the calculator!

Since we do not see a degree symbol, we must assume that we are dealing with radians.

Therefore, we will use the conversion $\left(\frac{180}{\pi}\right)^{\circ} \equiv 1$. We multiply both sides by $-\frac{4\pi}{3}$ to get the following:

$$-\frac{4\pi}{3}\left(\frac{180}{\pi}\right)^{\circ} \equiv -\frac{4\pi}{3}$$

Angle Measure in Radians

Example 5 continued:

Next, we reduce the fraction on the left of the equal sign to get the following:

$$-\frac{4\cancel{\pi}}{{}_{1}\cancel{3}}\left(\frac{\overset{60}{\cancel{180}}}{\cancel{\pi}}\right)^{\circ} \equiv -\frac{4\pi}{3}$$

We find that $-240^{\circ} \equiv -\frac{4\pi}{3}$.