Concepts
Solve Simple Trigonometric Equations

Based on power point presentations by Pearson Education, Inc.

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Learning Objectives

1. Solve simple trigonometric equations on a restricted solution interval.
2. Find the general solutions of simple trigonometric equations.

NOTE: This lesson contains some examples. You can find more examples in the “Examples” document also located in the appropriate MOM Learning Materials folder.

1. Solve Simple Trigonometric Equations on a Restricted Interval (1 of 14)

All trigonometric equations have infinitely many solutions! Often, the solutions are restricted to an interval in which case we may have no solutions or one or more solutions. Other times we are asked to find all solutions.

We will first learn how to solve simple trigonometric equations of the following forms all with solutions restricted to a given interval:

a* *sin*(*bx*) = *C

a* *cos*(*bx*) = *C

a* *tan*(*bx*) = *C

Note that ***b*** **= 1** and ***a*** and ***C*** are real numbers with $\mathbf{a \neq 0}$.

Solve Simple Trigonometric Equations on a Restricted Interval

Solution Strategy:

Step 1 - Examine the solution interval and keep it in mind. If necessary, isolate the trigonometric ratio. "Isolate" means that the trigonometric ratio must be the only term on one side of the equal sign and its coefficient must be 1.

Example 1:

Solve $\mathbf{2\ \mathit{sin}\ \mathit{x} = -1}$ for $\boldsymbol{x}$ in the interval $\mathbf{[0, 2\pi)}$. Express the solutions in EXACT radians.

The solution interval for angle $\boldsymbol{x}$ is $\mathbf{[0, 2\pi)}$. We will then isolate the trigonometric ratio by dividing both sides of the equal sign by 2.

We end up with $\sin x = -\frac{1}{2}$.

Solve Simple Trigonometric Equations on a Restricted Interval

Step 2 - We are going to use the concept of inverse trigonometric functions to solve for the angle ***x*** in the equation from Step 1 using the calculator.

Example 1 continued:

Given $\sin x = -\frac{1}{2}$, we find $x = \sin^{-1}\left(-\frac{1}{2}\right)$.

Helpful Hint: It is often easier to work in degrees and then change the solutions back to radians! So, let's change the solution interval to degrees, namely **[0°, 360°)**, and then we continue to work with degrees.

Therefore, with the calculator in degree mode, we get ***x* = – 30°**.

Solve Simple Trigonometric Equations on a Restricted Interval

Step 3 - If possible, find the reference angle for the solution in Step 2. Note, a reference angle will not exist if the solution in Step 2 is a quadrantal angle.

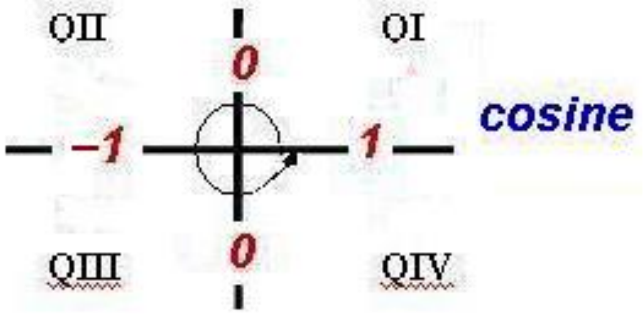
Example 1 continued:

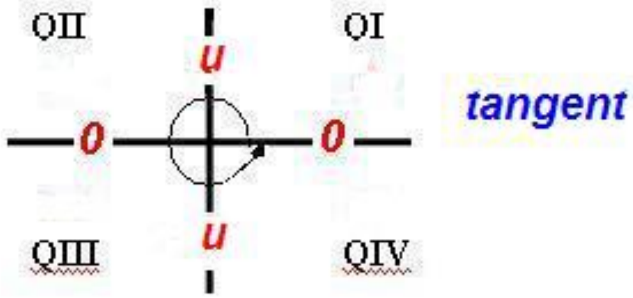
Given a solution for angle $\boldsymbol{x}$ in Step 2, we will find its reference angle. We know that the reference angle of a negative angle is equal to that of its positive counterpart. **Since 30^o is a QI angle, its reference angle equals 30^o. Therefore, the reference angle of -30^o is 30^o as well.**

Solve Simple Trigonometric Equations on a Restricted Interval

Step 4a - If a reference angle exists in Step 3 and knowing the sign of the trigonometric ratio from Step 1 (positive or negative), use this information in conjunction with *All Students Take Calculus* to find the solution(s) for angle ***x*** on the given solution interval!

Step 4b - If a reference angle does not exist in Step 3 (quadrantal angle), we must know the values of the trigonometric ratios of quadrantal angles to find the solution(s) for angle ***x*** on the given solution interval.



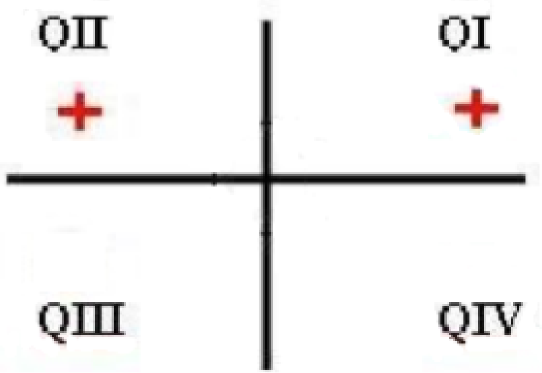


Solve Simple Trigonometric Equations on a Restricted Interval

Example 1 continued:

We have a reference angle and the value of the sine ratio in Step 1 is negative.

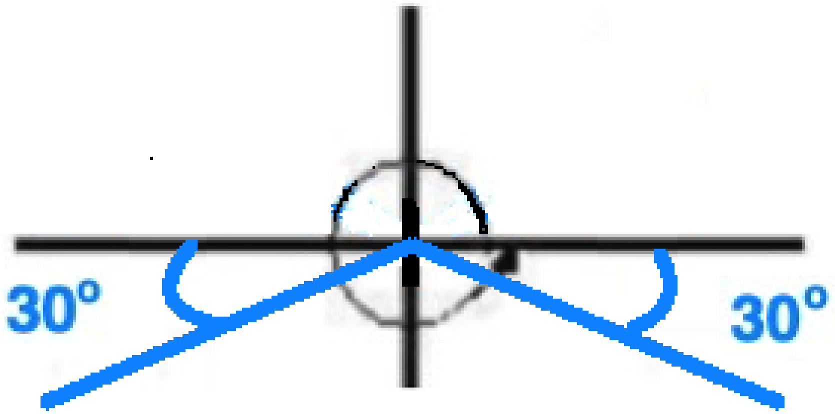
Therefore, we will use *All Students Take Calculus* to find the quadrants in which the sine ratio is positive. Thus, the sine is negative in QIII and QVI. See picture below!



Solve Simple Trigonometric Equations on a Restricted Interval

Example 1 continued:

Finally, let's first indicate the solution interval for angle ***x*** by graphing the angle **360º**. Then we draw the *reference angle* in the appropriate quadrants.



Solve Simple Trigonometric Equations on a Restricted Interval

Example 1 continued:

We can now find the solutions for angle $\boldsymbol{x}$ in the interval $\mathbf{[0^o, 360^o)}$ with the help of the picture on the previous slide.

Solution in QI: Solution in QII:

$\boldsymbol{x_1} = \mathbf{180^o + 30^o = 210^o}$ $\boldsymbol{x_2} = \mathbf{360^o - 30^o = 330^o}$

Since we are supposed to express the solutions in terms of radians, we simply change the degree solutions to radians to get the following:

Solution in QI: Solution in QII:

$\boldsymbol{x_1} = \frac{\mathbf{7\pi}}{\mathbf{6}}$ $\boldsymbol{x_2} = \frac{\mathbf{11\pi}}{\mathbf{6}}$

Solve Simple Trigonometric Equations on a Restricted Interval

Example 2:

Solve $\mathbf{2}\,\boldsymbol{cos\ x} = \mathbf{0}$ for $\boldsymbol{x}$ in the interval $\mathbf{[0^o, 360^o)}$. Express the solutions in EXACT degrees.

The solution interval is $\mathbf{[0^o, 360^o)}$. We will then isolate the trigonometric ratio by dividing both sides of the equal sign by 2.

We end up with $\boldsymbol{cos\ x} = \mathbf{0}$.

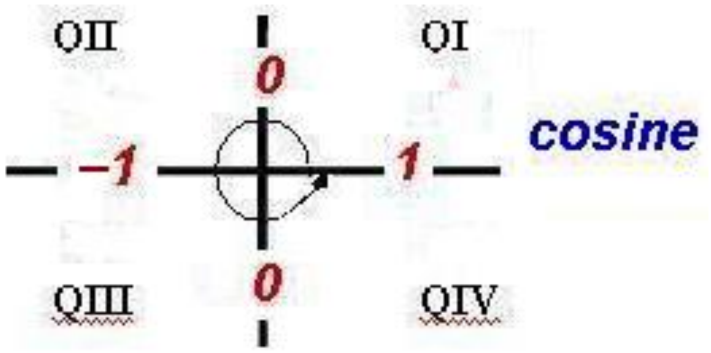
Now, we find the angle $\boldsymbol{x}$ by using the concept of inverse trigonometric functions:

$\boldsymbol{x = cos^{-1}}\mathbf{(0)}$

Solve Simple Trigonometric Equations on a Restricted Interval

Example 2 continued:

With the calculator in degree mode, we get $\boldsymbol{x = 90^o}$ which is a quadrantal angle. It does not have a reference angle! Therefore, we will use the following picture:



From the picture above, we find that ***cos x*** equals **0** at $\mathbf{90^o}$ and $\mathbf{270^o}$ in the interval $\mathbf{[0^o, 360^o)}$.

Therefore, we have two solutions, namely $\boldsymbol{x_1 = 90^o}$ and $\boldsymbol{x_2 = 270^o}$.

Solve Simple Trigonometric Equations on a Restricted Interval

Example 3:

Solve $\boldsymbol{\sin x = 1}$ for $\boldsymbol{x}$ in the interval **$[0^o, 360^o)$**. Express the solutions in EXACT degrees.

The solution interval is **$[0^o, 360^o)$**. The trigonometric ratio is already isolated.

Now, we find the angle $\boldsymbol{x}$ by using the concept of inverse trigonometric functions:

$$\boldsymbol{x = \sin^{-1}(1)}$$

Solve Simple Trigonometric Equations on a Restricted Interval

Example 3 continued:

With the calculator in degree mode, we get ***x* = 90º** which is a quadrantal angle. It does not have a reference angle! Therefore, we will use the following picture:

From the picture above, we note that ***sin x* = 1** at **90º** . This is the solution of ***sin x = 1*** in the interval **[0º, 360º)**.

Solve Simple Trigonometric Equations on a Restricted Interval

Example 4:

Solve $\boldsymbol{3\,sin\ x = 0}$ for $\boldsymbol{x}$ in the interval **[0°, 360°)**. Express the solutions in EXACT degrees.

The solution interval is **[0°, 360°)**. We will then isolate the trigonometric ratio by dividing both sides of the equal sign by 3.

We end up with $\boldsymbol{sin\ x = 0}$.

Now, we find the angle x by using the concept of inverse trigonometric functions:

$$\boldsymbol{x = sin^{-1}(0)}$$

Solve Simple Trigonometric Equations on a Restricted Interval

Example 4 continued:

With the calculator in degree mode, we get $\boldsymbol{x = 0^o}$ which is a quadrantal angle. It does not have a reference angle! Therefore, we will use the following picture:

From the picture above, we note that ***sin x*** equals **0** at **0º** and **180º** in the interval **[0º, 360º)**. Please note that **360º** is not a because the solution interval excludes this angle since there is a parenthesis next to it **360º** !!!

2. Find the General Solutions of Simple Trigonometric Equations (1 of 4)

Now we will learn how to find the general solutions of simple trigonometric equations of the following forms:

$$a \sin(bx) = C$$

$$a \cos(bx) = C$$

$$a \tan(bx) = C$$

Note that $b = 1$ and a and C are real numbers with $\mathbf{a \neq 0}$.

Find the General Solutions of Simple Trigonometric Equations (2 of 4)

Strategy for Finding the General Solutions:

Step 1 - Find the solutions in radians on the interval **[0, 2π)** or in degrees on the interval **[0^o, 360^o)**.

Example 5:

Find the general solution for the equation **2 *sin x* = 1**. Express it in EXACT radians.

In Example 1, we found that the EXACT solutions of the given equation in radians on the interval **[0, 2π)** are $\boldsymbol{x_1 = \frac{\pi}{6}}$ and $\boldsymbol{x_2 = \frac{5\pi}{6}}$.

Find the General Solutions of Simple Trigonometric Equations (3 of 4)

Step 2a - If the solutions in Step 1 are NOT exactly $180^o \equiv \pi$ apart, add $\mathbf{360^o k}$ or $\mathbf{2\pi k}$ to each of the solutions, where $\boldsymbol{k}$ is considered to be any integer. This is considered the general solution.

Step 2b - If the solutions in Step 1 are exactly $180^o \equiv \pi$ apart, add $\mathbf{180^o k}$ or $\boldsymbol{\pi}\mathbf{k}$ to the smaller solution only. Again, $\boldsymbol{k}$ is considered to be any integer. This is considered the general solution.

Example 5 continued:

Given $\boldsymbol{x_1} = \frac{\pi}{6}$ $(\equiv \mathbf{30^o})$ and $\boldsymbol{x_2} = \frac{5\pi}{6}$ $(\equiv \mathbf{150^o})$, we notice that the solutions are less than π $(\equiv \mathbf{180^o})$ apart.

Therefore, the general solution is $\frac{\pi}{6} + \mathbf{2\pi k}$ and $\boldsymbol{x_2} = \frac{5\pi}{6} + \mathbf{2\pi k}$.

Find the General Solutions of Simple Trigonometric Equations (4 of 4)

Example 6:

Find the general solution for the equation $\mathbf{2}\,\boldsymbol{\cos x} = \mathbf{0}$. Express it in EXACT degrees.

In Example 2, we found that the solutions of the given equation in degrees on the interval $\mathbf{[0^{o}, 360^{o})}$ are $\boldsymbol{x_1} = \mathbf{90^{o}}$ and $\boldsymbol{x_2} = \mathbf{270^{o}}$.

We notice that the solutions are exactly 180^{o} apart. Therefore, the general solution is $\boldsymbol{x} = \mathbf{90^{o}} + \mathbf{180^{o}k}$.

Note that we only used the smaller of the two solutions! When we let $\boldsymbol{k} = \mathbf{1}$, then we get the larger of the two solutions. Of course, $\boldsymbol{k}$ can be any other integer as well.