Concepts
Introduction to Parametric Equations

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

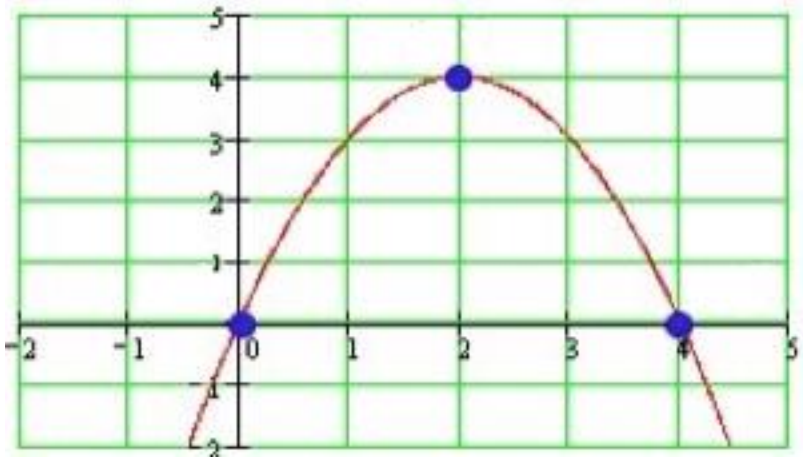
Learning Objectives

1. Parameterize functions.
2. Parameterize circles and ellipses.
3. Change parametric equations to rectangular equations.

NOTE: This lesson contains some examples. You can find more examples in the “Examples” document also located in the appropriate MOM Learning Materials folder.

Up to this point, we have been representing the graphs of equations in two variables by a single equation involving two variables, such as ***x*** and ***y*** or ***r*** and $\boldsymbol{\theta}$. Now we will study a few situations in which we will describe equations in two variables in a different way.

To start out, let's look at the equation $\boldsymbol{y = -x^2 + 4x}$. From algebra we know that this is a quadratic function, and its graph is a **parabola open down**.



What this graph does not tell us is the direction of movement of an object following the path of this parabola. Is the object traveling from the left to the right along the curve or is it traveling from the right to the left?

To determine the movement of an object along a curve, we will introduce an additional variable, called **parameter**. Using this parameter, we will then create separate equations, one for each variable in the given equation which created the curve. We call this **parameterizing** an equation in two or more variables.

As we know, there exist infinitely many equations in two or more variables. Each and every one can be parameterized. However, in this lesson we will only discuss a few functions as well as circles and ellipses (which are not functions) with center at the origin.

1. Parameterizing Functions (1 of 3)

Following is the strategy of parameterizing a few functions in *x* and *y*.

1. Decide on the name of the parameter. Most commonly in mathematics it involves the letter $\boldsymbol{t}$.

   We can just use $\boldsymbol{t}$ or any other expression, such as $\boldsymbol{t+1}$ or $\boldsymbol{t^2}$ etc. What we use for the parameter depends on the desired direction of movement. Note that we can find infinitely many sets of parametric equations for every function!!!

2. Replace the *x*-variable with the parameter. This is our first parametric equation. It actually represents the *x*-coordinates of all the points on the graph of function.

3. Express the *y*-variable in terms of the parameter. That is, replace the *x* variable in the function with the parameter. This is our second parametric equation. It represents the *y*-coordinates of all the points on the graph of the function.

Parameterizing Functions (2 of 3)

Example 1:

Find a set of parametric equations for the function $\boldsymbol{y = 1 - x^2}$.

Step 1: Let's use the parameter $\boldsymbol{-t}$. This is exclusively our choice!!!

Step 2: Let the variable *x* in the function equal $\boldsymbol{-t}$. That is, we get

$\boldsymbol{x = -t}$. This is the first parametric equation.

Step 3: Express *y* in terms of the parameter. That is, replace the *x* in the function with $\boldsymbol{-t}$.

Given $\boldsymbol{y = 1 - x^2}$, we write $\boldsymbol{y = 1 - (-t)^2}$ and finally $\boldsymbol{y = 1 - t^2}$. This is the second parametric equation.

This can also be expressed as the ordered pair $\boldsymbol{(-t, 1 - t^2)}$ where $\boldsymbol{-t}$ represents the *x*-coordinates of all points on the graph of the function and $\boldsymbol{1 - t^2}$ represents the *y*-coordinates.

Parameterizing Functions (3 of 3)

Example 1 continued:

In summary, we found a SET of parametric equations for $\boldsymbol{y = 1 - x^2}$, namely

$\boldsymbol{x = -t}$ and $\boldsymbol{y = 1 - t^2}$.

This can also be expressed as the ordered pair $\boldsymbol{(-t, 1 - t^2)}$ where $\boldsymbol{-t}$ represents the *x*-coordinates of all points on the graph of the function and

$\boldsymbol{1 - t^2}$ represents the *y*-coordinates.

Investigate the direction of movement by graphing the set of parametric equations in desmos.com! For example, you would input $\boldsymbol{(-t, 1 - t^2)}$.

Desmos will give us a compound inequality, $0 \leq t \leq 1$ below the coordinates. Then replace $\boldsymbol{t}$ on the left with $\boldsymbol{-10}$ and on the right with increasingly larger numbers. Watch the curve develop from right to left. That is, if an object were to travel on the curve, it would travel from right to left.

2. Parameterizing Circles and Ellipses (1 of 4)

Following is the strategy of parameterizing **circles and ellipses with centers at the origin.** Please note that they are NOT functions! The strategy is entirely different!

1. If necessary, change the equation to standard form as discussed in the conic section lessons. In our case, this would be $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.
2. Use the *Pythagorean Identity* where ***t*** is the parameter. That is, $\boldsymbol{\sin^2(t) + \cos^2(t) = 1}$ which can also be written as $\boldsymbol{[\sin(t)]^2 + [\cos(t)]^2 = 1}$.
3. Let ***sin(t)*** or ***cos(t)*** either be equal to $\frac{x}{a}$ or $\frac{y}{b}$. Our decision depends on the desired direction of movement of an object around a circle or ellipse. Is it moving clockwise or counter-clockwise? Solve the equation for either *x* or *y.* This is the first parametric equation.
4. Depending on our decision in Step 3, we will now change the other term in the given circle or ellipse to either ***sin(t)*** or ***cos(t)*** and solve the equation for either *x* or *y.* This is the second parametric equation.

Parameterizing Circles and Ellipses (2 of 4)

Example 2:

Find a set of parametric equations for the ellipse $\boldsymbol{4x^2 + 9y^2 = 36}$. Let the parameter be $\sin t = \frac{x}{3}$.

Step 1: Change the equation to standard form. Let's first divide both sides by 36 and reduce right away. We get the following:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

This can also be written as $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ using the *Power-of-a-Quotient Rule* from algebra.

Parameterizing Circles and Ellipses (3 of 4)

Example 2 continued:

Step 2: We will now use the *Pythagorean Identity* where ***t*** is the parameter. That is, $\boldsymbol{sin^2(t) + cos^2(t) = 1}$ which can also be written as $\boldsymbol{[sin(t)]^2 + [cos(t)]^2 = 1}$.

Step 3: We are asked to let $\sin t = \frac{x}{3}$.

Expressed in terms of *x*, this is ***x = 3sin(t)****.* This is the first parametric equation.

Step 4: Since we used $\sin t = \frac{x}{3}$, we will now use $\cos t = \frac{y}{2}$.

Expressed in terms of *y*, this is ***y = 2cos(t).*** This is the second parametric equation.

In summary, we found a SET of parametric equations for $\boldsymbol{4x^2 + 9y^2 = 36}$, namely ***x = 3sin(t)*** and ***y = 2cos(t)****.*

Parameterizing Circles and Ellipses (4 of 4)

Example 2 continued:

In summary, we found a SET of parametric equations for $4x^2 + 9y^2 = 36$, namely $x = 3\sin(t)$ and $y = 2\cos(t)$.

This can also be expressed as the ordered pair $(3\sin(t), 2\cos(t))$ where $3\sin(t)$ represents the *x*-coordinates of all points on the graph of the function and $2\cos(t)$ represents the *y*-coordinates.

Investigate the direction of movement by graphing the set of parametric equations in *desmos.com*! For example, you would input $(3\sin(t), 2\cos(t))$.

Desmos will give is a compound inequality, $0 \leq t \leq 1$ below the coordinates. Leave the ***0*** on the left and on the right use increasingly larger numbers. Watch the curve develop in a clockwise direction. That is, if an object were to travel on the curve, it would travel in a clockwise direction.

1. Change Parametric Equations to Rectangular Equations (1 of 4)

We are going to discuss two types of parametric equations, namely those that DO NOT contain trigonometric ratios and those that contain sine and cosine ratios.

1. Following is the strategy for finding rectangular equations **given parametric equations that DO NOT contain trigonometric ratios**.

Step 1 - Isolate the parameter in one of the two parametric equations.

NOTE: "Isolate" means that the parameter must be on one side of the equation by itself and must have a coefficient of 1.

Step 2 - Substitute the resulting expression in Step 1 for the parameter in the "other" parametric equation. Then simplify if necessary or desired.

Change Parametric Equations to Rectangular Equations

(2 of 4)

Example 3:

Find a rectangular equation in *x* and *y* given the set of parametric equations $x = t^2 - 4$ and $y = \frac{t}{2}$.

Step 1 - Isolate the parameter ***t*** in one of the parametric equations.

Let's use $y = \frac{t}{2}$ and solve for ***t***. We get ***2y = t***.

Step 2 - Substitute the ***2y*** from Step 1 for ***t*** in the other equation $x = t^2 - 4$.

We get $x = (2y)^2 - 4$ or $x = 4y^2 - 4$. From our study of conic sections we should know that this is the equation of a parabola open to the right with vertex at the origin!

Change Parametric Equations to Rectangular Equations

2. Following is the strategy for finding rectangular equations **given parametric equations containing sine and cosine ratios**.

Step 1 - Isolate the sine and cosine ratios on one side of the parametric equations.

NOTE: “Isolate” means that the parameter must be on one side of the equation by itself and must have a coefficient of 1.

Step 2 - Replace ***cos(t)*** and ***sin(t)*** in the *Pythagorean Identity* ***$sin^2(t) + cos^2(t) = 1$*** which can also be written as ***$[sin(t)]^2 + [cos(t)]^2 = 1$*** with the rectangular side of the equation in Step 1.

Change Parametric Equations to Rectangular Equations

(4 of 4)

Example 4:

Find a rectangular equation in *x* and *y* given the set of parametric equations ***x = 3cos(t)*** and ***y = 4sin(t)***.

Step 1 - Isolate the sine and cosine ratios as follows:

$\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$

Step 2 - Replace ***cos(t)*** and ***sin(t)*** in the *Pythagorean Identity* ***[sin(t)]² + [cos(t)]² = 1*** as follows:

$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$ or $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$. From our study of conic sections we should know that this is the equation of an ellipse since the denominators are not equal. Its center is at the origin.