Concepts
Trigonometric Ratios of Multiples of "Special" Angles

Based on power point presentations by Pearson Education, Inc.
Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Find reference angles in degrees and radians of angles.
2. Positive and negative values of trigonometric ratios.
3. Find the EXACT values of trigonometric ratios of integer multiples of "special" angles without a calculator.

NOTE: This lesson contains some examples. You can find more examples in the “Examples” document also located in the appropriate MOM Learning Materials folder.

1. Find Reference Angles in Degrees and Radians (1 of 10)

In the previous lesson we found and memorized the EXACT values of trigonometric ratios of "special" angles 30^o, 45^o, and 60^o. Now we will find the EXACT values of trigonometric ratios of integer multiples of "special" angles.

Before we find the EXACT values of these trigonometric ratios, two discussions are necessary. The first one deals with **reference angles**. **Let's call them α (alpha).** They are always positive and acute (less than 90^o) angles that lie between the terminal side of some given angle and the horizontal axis in a rectangular coordinate system. They can be measured in degrees and radians.

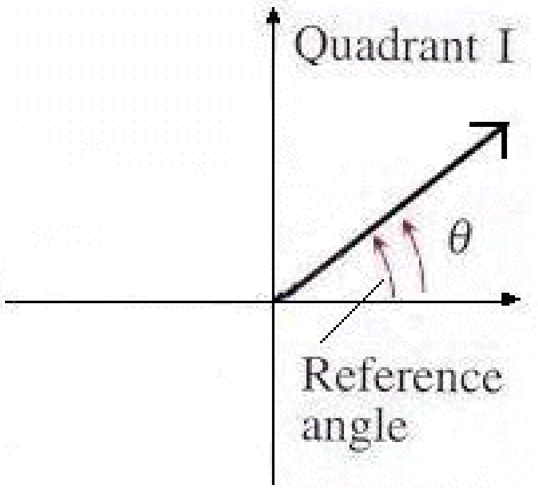
The second discussion reveals the fact that **values of trigonometric ratios can be positive and negative**.

Find Reference Angles in Degrees and Radians (2 of 10)

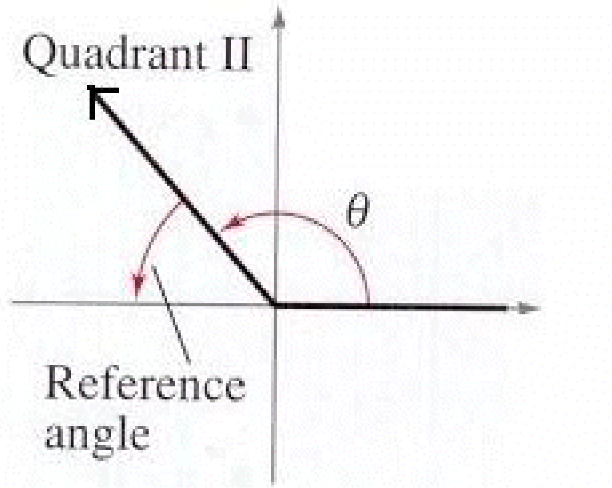
Strategy for finding *Reference Angles*. You MUST memorize the procedures and the reference angle calculations.

NOTE: For the strategy to work, the measure of a given angle θ MUST be between 0^o and 360^o (or 0 and 2π if the angle is in radian measure).

1. If θ is a Quadrant I angle, then the reference angle α equals θ.



2. If θ is a Quadrant II angle, then the reference angle α equals $180 - \theta$ or $\pi - \theta$.

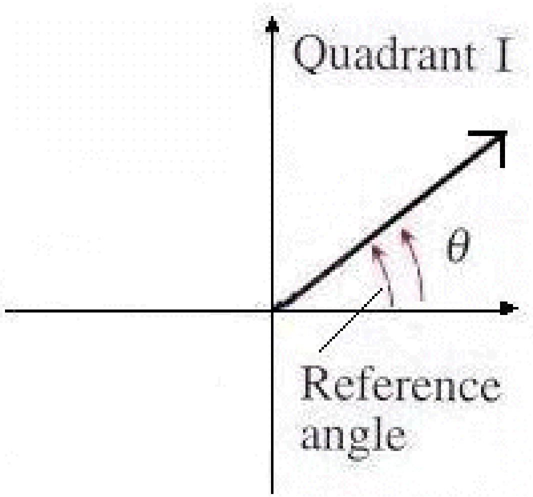


Find Reference Angles in Degrees and Radians (3 of 10)

Example 1:

Find the reference angle α for an angle of magnitude 75^o.

75^o is a QI angle (hint: graph it). Its reference angle equals 75^o as well.

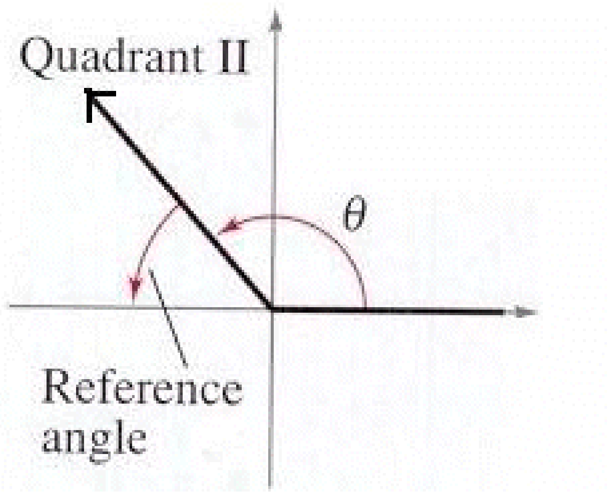


Find Reference Angles in Degrees and Radians

Example 2:

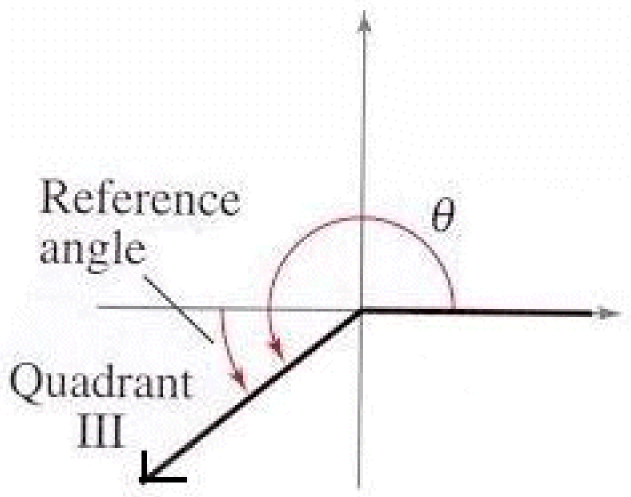
Find the reference angle α for an angle of magnitude 150^o.

150^o is a QII angle (hint: graph it). Its reference angle equals $180^o - 150^o = 30^o$.

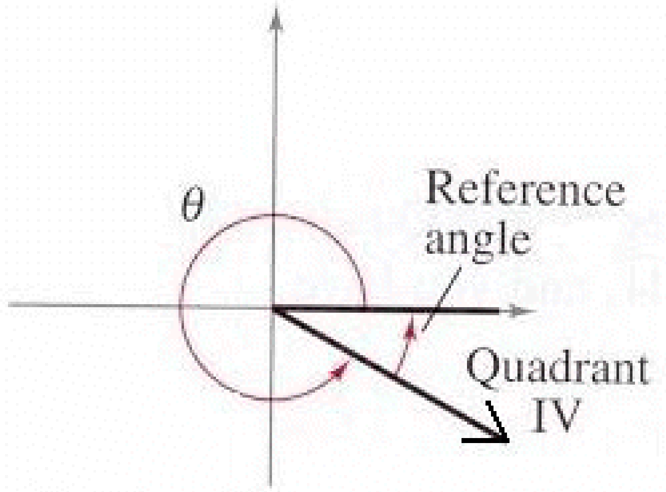


Find Reference Angles in Degrees and Radians (5 of 10)

3. If θ is a Quadrant III angle, then the reference angle α equals θ – 180º or θ – π.



4. If θ is a Quadrant IV angle, then the reference angle α equals 360 – θ or 2π – θ.

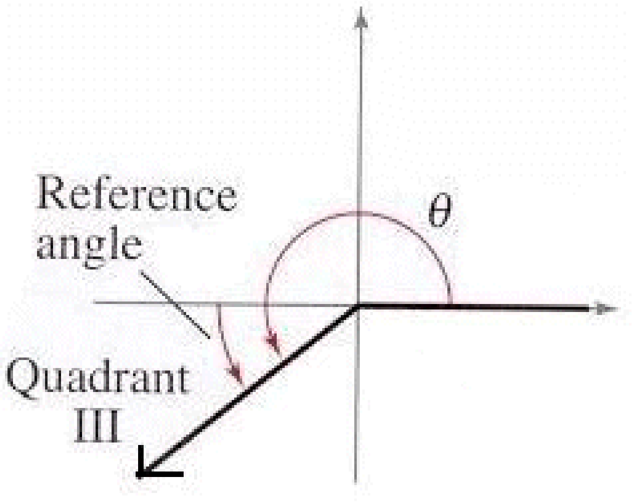


Find Reference Angles in Degrees and Radians (6 of 10)

Example 3:

Find the reference angle α for an angle of magnitude 225^o.

225^o is a QIII angle (hint: graph it). Its reference angle equals $225^o - 180^o = 45^o$.

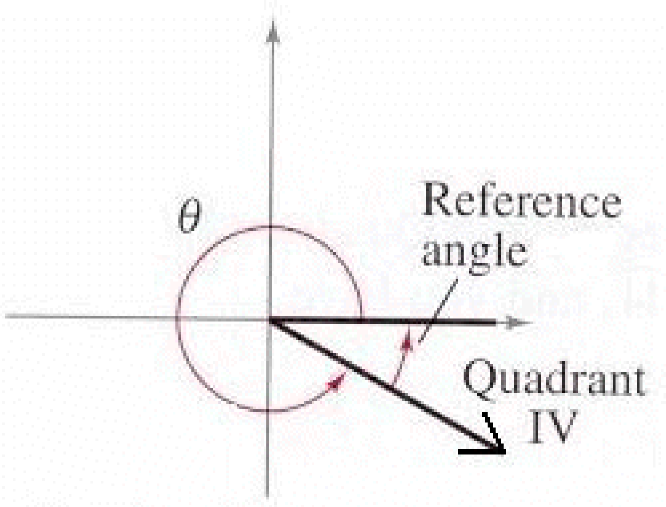


Find Reference Angles in Degrees and Radians (7 of 10)

Example 4:

Find the reference angle α for an angle of magnitude 300^o.

300^o it is a QIV angle (hint: graph it). Its reference angle equals $360^o - 300^o = 60^o$.



Find Reference Angles in Degrees and Radians (8 of 10)

Four Characteristics of Reference Angles. Memorize them!

1. The trigonometric ratio of an angle θ and the trigonometric ratio of its reference angle α have the same ABSOLUTE value.
2. Negative angles have the same reference angles as their positive counterparts.
3. Coterminal angles have the same reference angle.

Find Reference Angles in Degrees and Radians

Example 5:

Would the angle θ with measure 480^o considered to be an integer multiple of 30^o or 60^o?

We do know that $16(30^o) = 480^o$ and $8(60^o) = 480^o$. Here we are going to use the reference angle to help us decide if angle θ is considered an integer multiple of 30^o or 60^o.

Let's sketch this angle in a rectangular coordinate system.

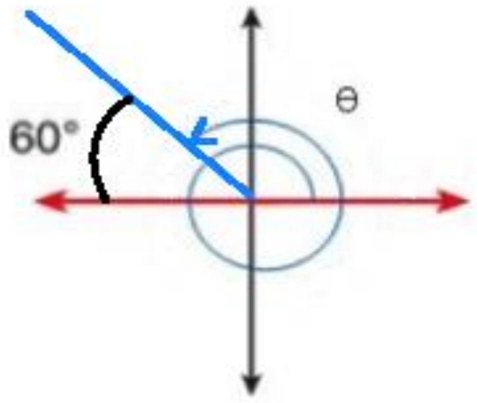
Before we do this, we always "take out" all of the full angles, namely 360^o and then look at the remainder.

Since $480^o = 360^o(1) + 120^o$, we will use the remainder 120^o to help us with the sketch. Specifically, we will sketch 360^o and then add 120^o to this arc.

Find Reference Angles in Degrees and Radians (10 of 10)

Example 5 continued:

Following is the sketch of 480^o in a rectangular coordinate system.



Now, we look at the reference angle of 120^o. It lies between its terminal side and the horizontal axis. The picture above shows that the reference angle is $60^o = 180^o – 120^o$. This tells use that 480^o is an integer multiple of 60^o.

2. Positive and Negative Values of Trigonometric Ratios
(1 of 5)

For the subsequent discussion, we will use a 30-60-90 triangle whose sides are in the proportions $c, c\sqrt{3}, 2c$ with ***2c*** being the length of the hypotenuse.

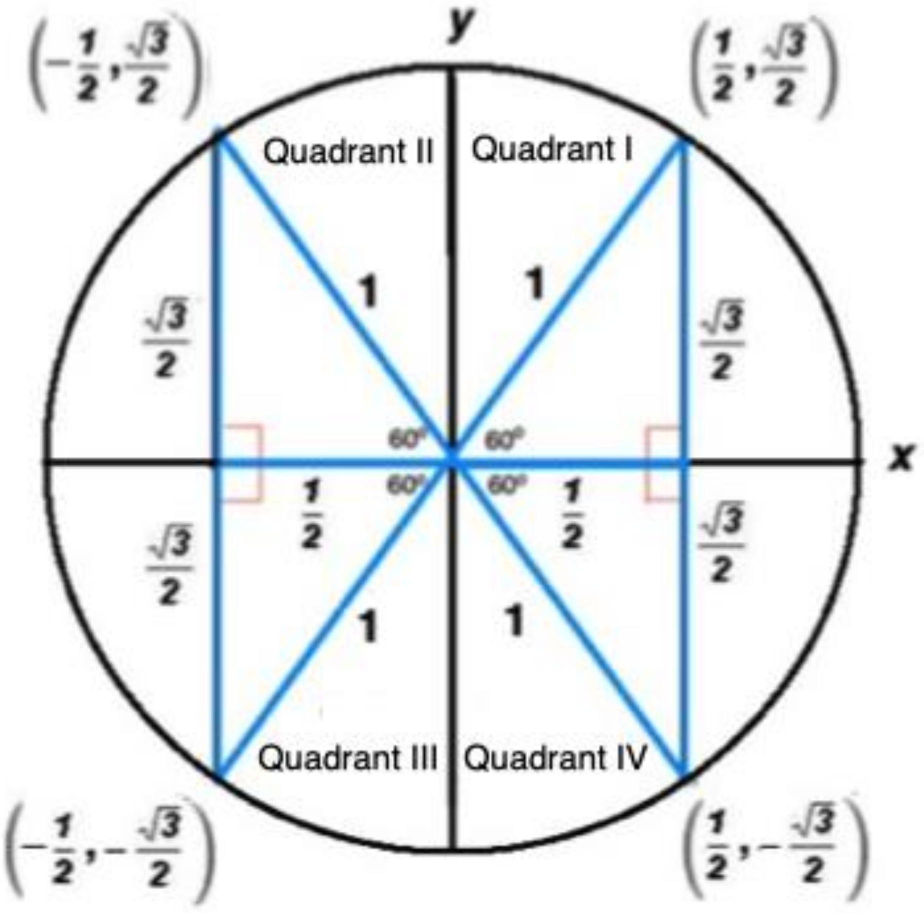
Specifically, we will draw right triangles containing 30º and 60º angles into each quadrant of a rectangular coordinate containing an *x*- and a *y*-axis. We let the 60º angle be next to the origin and opposite the 90º angle.

Furthermore, we will let $c = x = \frac{1}{2}$.

NOTE: We could have also used a 45-45-90 triangle for our discussion. All we want to show is that the values of trigonometric ratios of some angles are negative!

Positive and Negative Values of Trigonometric Ratios (2 of 5)

Following is a picture of what was described in the previous slide. Actually, we show a unit circle (discussed in the previous lesson). Note that the values of trigonometric ratios of the "special" angles 30°, 45°, and 60° (or other acute angles) are located in QI.



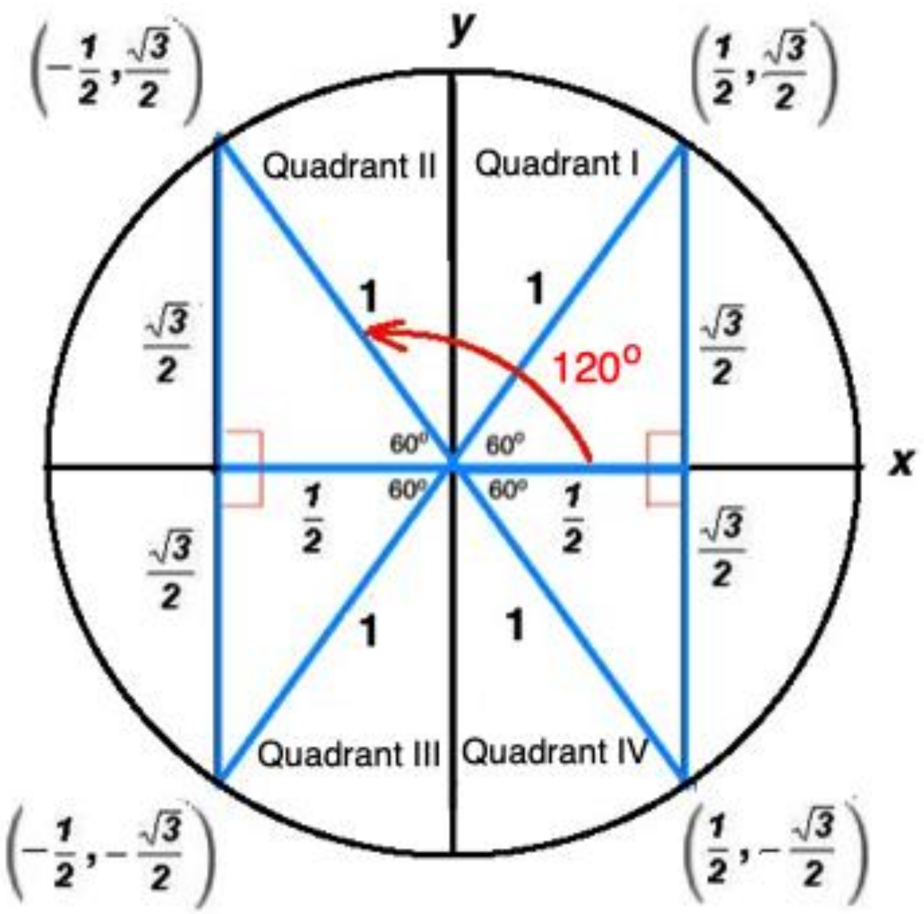
As we can see, these four triangles produce the points $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Now, instead of using the sides of a right triangle to define the values of the cosine, sine, and tangent ratios, we will use the four points stated above.

NOTE: Their ***x*-coordinates** will be equivalent to the sides adjacent **(adj)**, and their ***y*-coordinates** will be equivalent to the sides opposite **(opp)** the 60° angle.

Positive and Negative Values of Trigonometric Ratios (3 of 5)

Now let's look at angles that are larger than 90°. Specifically, let's use the angle of magnitude 120° to show that some of the values of trigonometric ratios are negative!



The initial side of the angle 120° is the positive *x*-axis, and the line through through the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in Quadrant II is its terminal side.

Using **adj** = ***x*** = $-\frac{1}{2}$ and **hyp** = ***1*** and using $\cos\theta = \frac{adj}{hyp}$ we find that **cos 120°** = $-\frac{1}{2}$.

Using **opp** = ***y*** = $\frac{\sqrt{3}}{2}$ and **hyp** = ***1*** and using $\sin\theta = \frac{opp}{hyp}$ we find that **sin 120°** = $\frac{\sqrt{3}}{2}$.

Using **opp** = ***y*** = $\frac{\sqrt{3}}{2}$ and **adj** = ***x*** = $-\frac{1}{2}$ and using $\tan\theta = \frac{opp}{adj}$ we find that **tan 120°** = $-\sqrt{3}$.

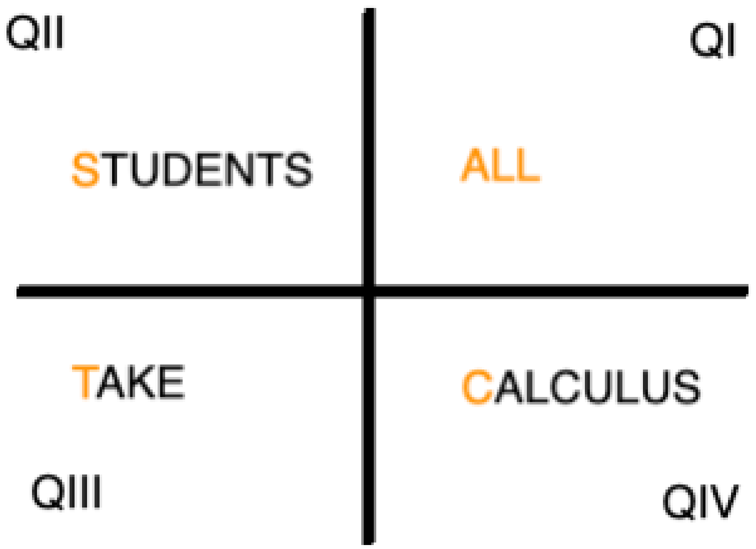
Positive and Negative Values of Trigonometric Ratios (4 of 5)

What we did for the angle of magnitude 120° on the previous slide, we can do with every angle. Eventually, we can establish the following pattern for any angle which must be memorized.

Terminal side of the angle is in Quadrant I in a rectangular coordinate system	All trigonometric ratios have positive values
Terminal side of the angle is in Quadrant II in a rectangular coordinate system	Sine (and cosecant) ratios ONLY have positive values
Terminal side of the angle is in Quadrant III in a rectangular coordinate system	Tangent (and cotangent) ratios ONLY have positive values
Terminal side of the angle is in Quadrant IV in a rectangular coordinate system	Cosine (and secant) ratios ONLY have positive values

Positive and Negative Values of Trigonometric Ratios (1 of 2)

Following is a handy memorization aid for the information in the table on the previous slide. We say, **"All Students Take Calculus"**. It lets us know in which quadrant the value of the trigonometric ratio of an angle is positive. Of course, the other values are negative.



The **ALL** is for all ratios have positive values.

The **S** in ”Students” is for SINE (and cosecant) ratios have positive values.

The **T** in ”Take” is for TANGENT (and cotangent) ratios have positive values.

The **C** in ”Calculus” is for COSINE (and secant) ratios have positive values.

3. Find the Values of Trigonometric Ratios of Integer Multiples of “Special” Angles (1 of 5)

Now, we are going to use the concepts of reference angles and the table indicating the positive and negative values of trigonometric ratios to create a strategy for finding values of trigonometric ratios of integer multiples of special angles in degrees and radians. NOTE: You MUST find the values of trigonometric ratios of integer multiples of special angles without a calculator in the homework and on the Module 1 Quiz.

Step 1 - Find the reference angle of a given angle θ. Use one or more of the four characteristics of reference angles.

Example 6:

Find the EXACT value of $\cos 300^{o}$ without a calculator.

300^{o} is a QIV angle (hint: graph it), therefore, its reference angle α is calculated as follows: $360^{o} – 300^{o} = 60^{o}$

Find the Values of Trigonometric Ratios of Integer Multiples of "Special" Angles (2 of 5)

Step 2 - Find the EXACT value of the trigonometric ratio of the reference angle found in Step 1.

Example 6 continued:

The reference angle is 60^o, and we memorized that $\cos 60^o = \frac{1}{2}$.

Step 3 - Find the EXACT value of the trigonometric ratio of the given angle θ with the help of the following:

a. The value of the trigonometric ratio of the reference angle found in Step 2.
b. One or more of the four characteristics of reference angles.
c. "All Students Take Calculus" to find if the value of the trigonometric ratio of the given angle θ is positive or negative.

Find the Values of Trigonometric Ratios of Integer Multiples of “Special” Angles (3 of 5)

Example 6 continued:

a. We use $\frac{1}{2}$ found in Step 2.

b. One of the four characteristics of reference angles states that the value of a trigonometric ratio of an angle θ and that of its reference angle α have the same ABSOLUTE value.

   Therefore, the value of cos 300^o is either $\frac{1}{2}$ or $-\frac{1}{2}$.

c. According to "All Students Take Calculus", the cosine ratio has positive values in QI and QIV. The terminal side of 300^o lies in QIV.

   Consequently, cos 300^o must have a positive value.

Specifically, cos 300^o = $\frac{1}{2}$.

Find the Values of Trigonometric Ratios of Integer Multiples of “Special” Angles

Example 7:

Find the EXACT value of sin (– 135^o) without a calculator.

We need to find the reference angle. However, the given angle is not between 0^o and 360^o. Therefore, we cannot use one of the four reference angle calculations we learned earlier.

However, one of the four characteristics of reference angles states that negative angles have the same reference angles as their positive counterparts. Therefore, we will find the reference angle using + 135^o.

135^o is a QII angle (hint: graph it), therefore, its reference angle α is calculated as follows: $180^o - 135^o = 45^o$

Find the Values of Trigonometric Ratios of Integer Multiples of “Special” Angles (5 of 5)

Example 7 continued:

The reference angle is 45°, and we memorized that sin 45° = $\frac{\sqrt{2}}{2}$.

We will use $\frac{\sqrt{2}}{2}$.

One of the four characteristics of reference angles states that the value of a trigonometric ratio of an angle θ and that of its reference angle α have the same ABSOLUTE value.

Therefore, the value of sin (– 135°) is either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$.

According to "All Students Take Calculus", the sine ratio has positive values in QI and QII. The terminal side of (– 135°) lies in QIII (hint: graph it). Consequently, sin (– 135°) must have a negative value.

Specifically, sin (– 135°) = $-\frac{\sqrt{2}}{2}$.