Concepts
The Laws of Sines and Cosines

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Solve oblique triangles using the Law of Sines.
2. Solve oblique triangles using the Law of Cosines.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

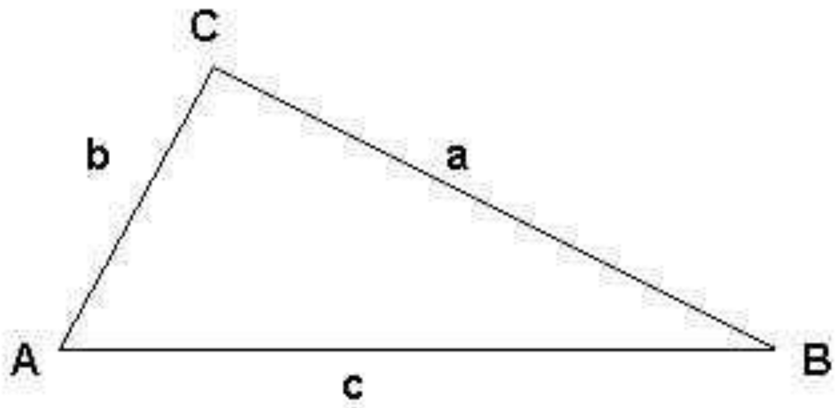
In a previous lesson, we learned how to use the trigonometric ratios and the *Pythagorean Theorem* to find missing sides and angles in a right triangle. We called this process "solving a triangle." Now, we will learn how to do the same for oblique triangles, that is, triangles that do NOT contain a right angle.

Some oblique triangles can be solved using the **Law of Sines** for others we must use the **Law of Cosines**. Incidentally, these laws can also be used with right triangles.

1. The Law of Sines (1 of 9)

To use the **Law of Sines**, the information given must include one angle and its opposite side. You do not have to memorize the *Law of Sines* however, you must know how to work with it when given.

Let's examine the picture below of a triangle with angles ***A***, ***B***, and ***C***, and corresponding opposite sides labeled ***a***, ***b***, and ***c*.**



The **Law of Sines** states the following:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Law of Sines (2 of 9)

The proof for the **Law of Sines** is provided separately. If you are interested, it can be found in the corresponding Learning Materials in the MyOpenMath course.

Please note that the **Law of Sines** sometimes produces TWO different sets of solutions!

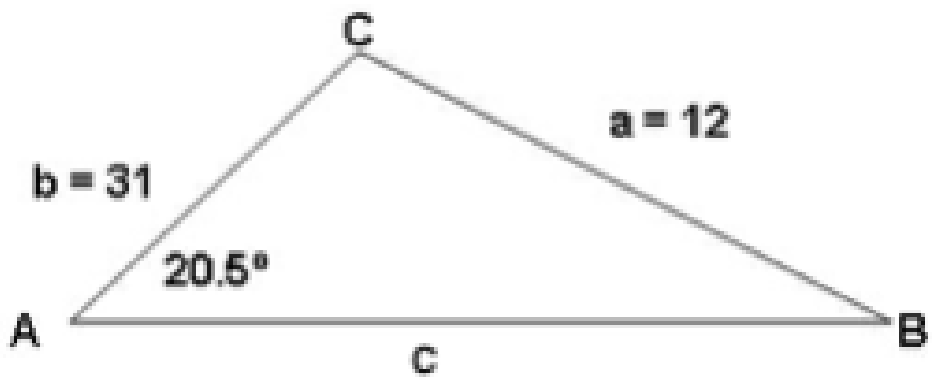
The Law of Sines (3 of 9)

Example 1:

For an oblique triangle with side ***a*** **= 12 ft**, side ***b*** **= 31 ft**, and angle ***A*** **= 20.5º**,

find the remaining sides and angles. DO NOT ROUND UNTIL THE VERY END OF EACH CALCLATION. Then round the final answer to two decimal places. Assume angles ***A***, ***B***, and ***C*** are opposite sides labeled ***a***, ***b***, and ***c*** respectively.

HINT: Make a sketch!



The Law of Sines (4 of 9)

Example 1 continued:

Since the information given includes an angle and its opposite side, we can try to use the **Law of Sines**.

Specifically, we will use $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Solve for Angle ***B***:

We are going to use two parts of the Law of Sines, namely, the one including angle ***A*** and ***B***, and sides ***a*** and ***b***. Specifically, we will write the following:

$$\frac{12}{\sin 20.5^\circ} = \frac{31}{\sin B}$$

The Law of Sines (5 of 9)

Example 1 continued:

Then **12 *sin B* = 31 *sin* 20.5º** and $\sin B = \dfrac{31 \sin 20.5^\circ}{12}$.

Now we must be very careful. Remember that we are only allowed to round the final answer. However, at this point we do want to know if ***sin B*** is positive or negative. Quickly using a calculator, we find that the ratio is positive.

Now, from *All Students Take Calculus* we know that the sine ratio is positive in QI and QII. Since the sum of interior angles in a triangle is 180º and QI and QII angles are less than 180º, we could possibly get two solutions for angle **B**, one acute and one obtuse angle.

NOTE: You always must check for two solution sets when using the Law of Sines!

The Law of Sines (6 of 9)

Example 1 continued:

If $\sin B = \dfrac{31\sin 20.5^\circ}{12}$, then $B = \arcsin\dfrac{31\sin 20.5^\circ}{12}$.

Using the calculator, we find $\boldsymbol{B} \approx \mathbf{64.78^o}$. This is a QI angle. What about a QII angle?

We will use the reference angle of $\mathbf{64.78^o}$ which is $\mathbf{64.78^o}$. Then the QII angle, let's call it $\boldsymbol{B_2}$**,** is approximately equal to $\mathbf{180^o - 64.78^o = 115.22^o}$.

Both angle $\boldsymbol{B}$ and $\boldsymbol{B_2}$ could be solutions for the given triangle as long as angle $\boldsymbol{C}$ does not become negative!

The Law of Sines (7 of 9)

Example 1 continued:

Solve for Angle ***C***:

We know that angle $\boldsymbol{A = 20.5^{o}}$, and we found that angle $\boldsymbol{B \approx 64.78^{o}}$ and angle $\boldsymbol{B_2 \approx 115.22^{o}}$.

Then angle ***C*** could be approximately equal to the following two measures:

$\boldsymbol{C \approx 180^{o} - 20.5^{o} - 64.78^{o} = 94.72^{o}}$

or

$\boldsymbol{C_2 \approx 180^{o} - 20.5^{o} - 115.22^{o} = 44.28^{o}}$

Since both measures for angle ***C*** are positive the triangle indeed has TWO solution sets.

The Law of Sines (8 of 9)

Example 1 continued:

Solve for side ***c***:

Here we need to find two measures for side ***c***. Namely, given angle $\boldsymbol{B} \approx \mathbf{64.78^o}$ and angle $\boldsymbol{C} \approx \mathbf{94.72^o}$. And also given angle $\boldsymbol{B_2} \approx \mathbf{115.22^o}$ and angle $\boldsymbol{C_2} \approx \mathbf{44.28^o}$.

We can now use the **Law of Sines** to find side ***c*** and side c_2 as follows:

$$\frac{12}{\sin 20.5^\circ} = \frac{c}{\sin 94.72^\circ} \quad \text{and} \quad \frac{12}{\sin 20.5^\circ} = \frac{c_2}{\sin 44.28^\circ}$$

$$c = \frac{12 \sin 94.72^\circ}{\sin 20.5^\circ} \approx 34.15 \quad \text{and} \quad c_2 = \frac{12 \sin 44.28^\circ}{\sin 20.5^\circ} \approx 23.92$$

The Law of Sines (9 of 9)

Example 1 continued:

In summary, the triangle with side $\boldsymbol{a}$ **= 12 ft**, side $\boldsymbol{b}$ **= 31 ft**, and angle $\boldsymbol{A}$ **= 20.5°** has the following two solutions:

angle $\boldsymbol{B} \approx \mathbf{64.78^o}$ and angle $\boldsymbol{C} \approx \mathbf{94.72^o}$ and side $\boldsymbol{c} \approx \mathbf{34.15\ ft}$

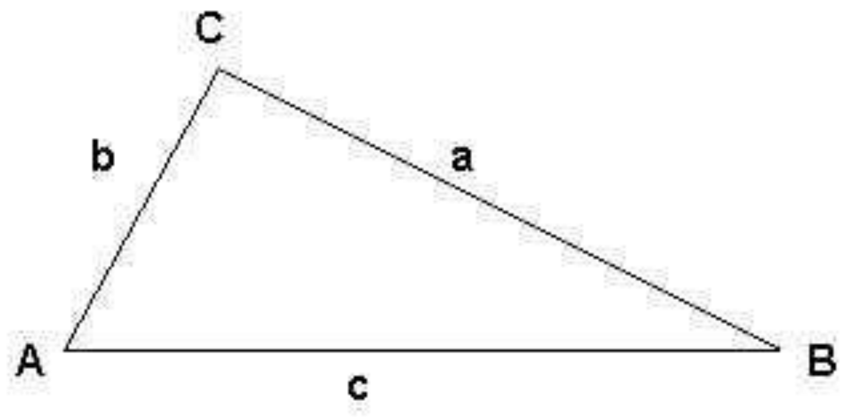
or angle $\boldsymbol{B_2} \approx \mathbf{115.22^o}$ and angle $\boldsymbol{C_2} \approx \mathbf{44.28^o}$ and side $\boldsymbol{c_2} \approx \mathbf{23.92\ ft}$

2. The Law of Cosines (1 of 9)

When the *Law of Sines* cannot be applied, we must use the **Law of Cosines**. You do not have to memorize the *Law of Cosines* however, you must know how to work with it when given.

NOTE: Once we start using this law, it is best to continue to use it for all calculations even if we could switch to the *Law of Sines* eventually.

Let's examine the picture below of a triangle with angles ***A***, ***B***, and ***C***, and corresponding opposite sides labeled ***a***, ***b***, and ***c***.



The **Law of Cosines** states the following:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

The Law of Cosines (2 of 9)

Observing the pattern of this formula, we can also write the following:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

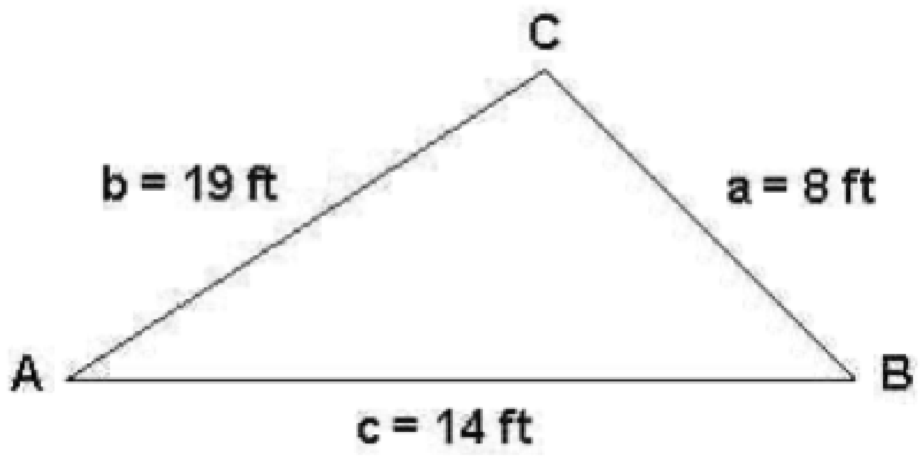
The proof for the **Law of Cosines** is provided separately. If you are interested, it can be found in the corresponding Learning Materials in the MyOpenMath course.

The Law of Cosines (3 of 9)

Example 2:

For an oblique triangle with side ***a* = 8 ft**, side ***b* = 19 ft**, and side ***c* = 14 ft**, find the angles. DO NOT ROUND UNTIL THE VERY END OF EACH CALCLATION. Then round the final answer to two decimal places. Assume angles ***A***, ***B***, and ***C*** are opposite sides labeled ***a***, ***b***, and ***c*** respectively.

HINT: Make a sketch!



The Law of Cosines (4 of 9)

Example 2 continued:

Since the information given does NOT include an angle and its opposite side, we cannot use the *Law of Sines*. Therefore, let's try the **Law of Cosines**.

Solve for Angle ***A***:

We are going to use $a^2 = b^2 + c^2 - 2bc \cos A$. Specifically, we will write the following:

$$8^2 = 19^2 + 14^2 - 2(19)(14) \cos A$$

Using the calculator, we find $64 = 557 - 532 \cos A$.

Isolating $\cos A$ on one side of the equal sign, we get $\cos A = \frac{493}{532}$.

The Law of Cosines

Example 2 continued:

Then $A = \cos^{-1}\left(\frac{493}{532}\right)$.

Note that 493/532 was not changed to a decimal approximation in the interest of more exact solutions.

Using the calculator, we find $\boldsymbol{A \approx 22.08^{\circ}}$.

There cannot be another solution because the only other quadrant in which the cosine ratio is positive is QIV, but there the angles are greater than the sum of the interior angles of a triangle.

The Law of Cosines (6 of 9)

Example 2 continued:

Solve for Angle ***B***:

This time we are going to use $b^2 = a^2 + c^2 - 2ac \cos B$. Specifically, we will write the following:

$$19^2 = 8^2 + 14^2 - 2(8)(14) \cos B$$

Using the calculator, we find $361 = 260 - 224 \cos B$.

Isolating ***cos B*** on one side of the equal sign, we get $\cos B = -\frac{101}{224}$.

The Law of Cosines (7 of 9)

Example 2 continued:

Then $B = \cos^{-1}\left(-\frac{101}{224}\right)$.

Note that – 101/224 was not changed to a decimal approximation in the interest of more exact solutions.

Using the calculator, we find $\boldsymbol{B} \approx \mathbf{116.80^{o}}$.

There cannot be another solution because the only other quadrant in which the cosine ratio is negative is QIII, but there the angles are greater than the sum of the interior angles of a triangle.

The Law of Cosines (8 of 9)

Example 2 continued:

Solve for Angle ***C***:

We found that angle $A \approx \mathbf{22.08^o}$ and angle $B \approx \mathbf{116.80^o}$. To ensure that the measures of all angles add up to 180^o, we will now do a simply subtraction.

Then angle ***C*** is approximately equal to the following:

$$C \approx \mathbf{180^o - 22.08^o - 116.80^o = 41.12^o}$$

The Law of Cosines (9 of 9)

Example 2 continued:

In summary, the triangle with side $\boldsymbol{a}$ **= 8 ft**, side $\boldsymbol{b}$ **= 19 ft**, and side $\boldsymbol{c}$ **= 14 ft**, has the following solutions:

angle $\boldsymbol{A} \approx \mathbf{22.08^o}$, angle $\boldsymbol{B} \approx \mathbf{116.80^o}$, and angle $\boldsymbol{C} \approx \mathbf{41.12^o}$