Concepts
Solve More Complex Trigonometric Equations

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Solve equations containing squared trigonometric ratios.
2. Solve simple trigonometric equations involving multiples of an angle.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

In this lesson, we will solve more complex trigonometric equations, namely those that contain squared trigonometric ratios and those that contain multiples of an angle.

From algebra you should remember that quadratic equations have squared variables, and we often solve them using the *Quadratic Formula*.

As a reminder, the *Quadratic Formula* states the following:

Given the general form of the quadratic equation $ax^2 + bx + c = 0$, its solutions for ***x*** are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The symbol $\pm$ indicates that there could potential be two solutions. One involving a negative square root and the other one a positive square root!

Note that the formula requires *a*, *b*, and *c* from the general form.

From algebra you should also remember that quadratic equations only containing a squared term, can be solved using the *Square Root Property* instead of the *Quadratic Formula*.

As a reminder, the *Square Root Property* states the following:

Given a quadratic equation of the form $\boldsymbol{u^2 = d}$, its solutions are $\boldsymbol{u = \pm\sqrt{d}}$. Please note that we have two solutions, namely $\boldsymbol{u = \sqrt{d}}$ and $\boldsymbol{u = -\sqrt{d}}$.

1. Solve Equations containing Squared Trigonometric Ratios.

First, we will learn how to solve equations containing squared trigonometric ratios.

Solution Strategy:

Step 1 - If possible, change all trigonometric ratios in the equation to the same ratio. That is, the equation must contain all sines, cosines, tangents, etc. Apply the appropriate identities to accomplish this! If this seems not possible, try factoring out the greatest common factor.

Solve Equations containing Squared Trigonometric Ratios.

Example 1:

Solve $\mathbf{2\,\boldsymbol{cos}^2\,\boldsymbol{x} + 3\,\boldsymbol{sin}\,\boldsymbol{x} = 0}$ for $\boldsymbol{x}$ on the interval $\mathbf{[0^o, 360^o)}$. Express the solutions in degrees.

Here we will try to change all trigonometric ratios to the same ratio. Remembering the *Pythagorean Identity* $\boldsymbol{cos^2\,x = 1 - sin^2\,x}$, we can change $\boldsymbol{2\,cos^2\,x + 3\,sin\,x = 0}$ to the following:

$$\mathbf{2\,(1 - \boldsymbol{sin}^2\,\boldsymbol{x}) + 3\,\boldsymbol{sin}\,\boldsymbol{x} = 0}$$

$$\mathbf{2 - 2\,\boldsymbol{sin}^2\,\boldsymbol{x} + 3\,\boldsymbol{sin}\,\boldsymbol{x} = 0}$$

Note that his equation is "quadratic" in form. Here the variables are ***sin x***, not just ***x*** !!!

Solve Equations containing Squared Trigonometric Ratios.

Step 2 – Use the *Quadratic Formula* or the *Square Root Property* or *factoring* to find one or more “simple trigonometric equations” as presented in the previous lesson.

Example 1 continued:

To make the solution process a little bit more palatable, let’s use the following substitution:

Let $\boldsymbol{\sin x = k}$, then $\boldsymbol{\sin^2 x = k^2}$.

It follows that $\mathbf{2 - 2\sin^2 x + 3\sin x = 0}$ becomes $\mathbf{2 - 2k^2 + 3k = 0}$ or $\mathbf{-2k^2 + 3k + 2 = 0}$.

Solve Equations containing Squared Trigonometric Ratios.

Example 1 continued:

We are now going to use the *Quadratic Formula* to solve $\mathbf{2 - 2k^2 + 3k = 0}$ for $\boldsymbol{k}$.

For this procedures, we notice that $\boldsymbol{a} = \mathbf{-2}$, $\boldsymbol{b} = \mathbf{3}$, and $\boldsymbol{c} = \mathbf{2}$.

Then $k = \dfrac{-(3) \pm \sqrt{3^2 - 4(-2)(2)}}{2(-2)}$

and $k = \dfrac{-3 \pm \sqrt{25}}{-4} = \dfrac{-3 \pm 5}{-4}$.

Solve Equations containing Squared Trigonometric Ratios.

Example 1 continued:

We find two solutions, namely $k = \frac{-3+5}{-4} = -\frac{1}{2}$ and $k = \frac{-3-5}{-4} = 2$.

Since ***k*** is actually equal to ***sin x***, we will back-substitute to get

$\sin x = -\frac{1}{2}$ and $\sin x = 2$ which are two simple trigonometric equations.

Steps 3 through 6 - Find the solutions for the angle ***x*** in all simple trigonometric equations found in Step 2 on the given solution interval. Use Steps 1 through 4a or 4b from the previous lesson on solving simple trigonometric equations.

Solve Equations containing Squared Trigonometric Ratios.

Example 1 continued:

Using Steps 1 through 4a and 4b from the previous lesson, we find the solutions for angle ***x*** in the interval **[0º, 360º)**.

From $\sin x = -\frac{1}{2}$, we find the following solutions:

Solution in QI: Solution in QII:

$x_1 = 180^o + 30^o = 210^o$ $x_2 = 360^o - 30^o = 330^o$

See Example 1 in the concept lecture "Solve Simple Trigonometric Equations".

Solve Equations containing Squared Trigonometric Ratios.

Example 1 continued:

From $\sin x = 2$ we find the following solution:

$\boldsymbol{x_3}$ **= Domain Error**

Why? Please note that ***sin x*** is never greater than 1 (remember its graph), therefore ***sin x = 2*** is undefined.

In summary, the equation $\mathbf{2\cos^2 x + 3\sin x = 0}$ has two solutions for ***x*** on the interval $\mathbf{[0^o, 360^o)}$, namely $\mathbf{210^o}$ and $\mathbf{330^o}$.

2. Solve Simple Trigonometric Equations involving Multiples of an Angle (1 of 12)

We will now learn how to solve simple trigonometric equations of the following forms however, unlike in the previous lesson, ***b*** will NOT be equal to 1.

$$a \sin(bx) = C$$

$$a \cos(bx) = C$$

$$a \tan(bx) = C$$

Note that $b \neq 1$ and a and C are real numbers with $a \neq 0$.

Solve Simple Trigonometric Equations involving Multiples of an Angle (2 of 12)

The solution strategy is very similar to the one we we learned in the previous lesson “Solve Simple Trigonometric Equations” in which ***b*** was equal to **1**.

Step 1 - Change the solution interval for the angle ***x*** to the solution interval for angle ***bx***.

Example 2:

Solve **2 *cos 4x* = – 1** for ***x*** in the interval **[0, π)**. Express the solutions in EXACT radians.

We must find a solution interval for angle ***4x***. The given solution interval is for angle ***x*** and it is **[0, π)**. Then the solution interval for angle ***4x*** is four times as large, namely **[4(0), 4(π))** or **[0, 4π)**.

Solve Simple Trigonometric Equations involving Multiples of an Angle (3 of 12)

Step 2 through 5 - Work Steps 1 through 4 from the previous lesson "Solve Simple Trigonometric Equations" using the angle ***bx*** and its solution interval.

Example 2 continued:

Next, we will isolate the trigonometric ratio by dividing both sides of the equal sign by 2.

We end up with $\cos 4x = -\frac{1}{2}$.

Solve Simple Trigonometric Equations involving Multiples of an Angle

Example 2 continued:

We are going to use the concept of inverse trigonometric functions to solve for the angle ***4x*** in the equation from Step 1 using the calculator.

Given $\cos 4x = -\frac{1}{2}$, we find $4x = \cos^{-1}\left(-\frac{1}{2}\right)$.

Helpful Hint: It is often easier to work in degrees and then change the solutions back to radians! So, let's change the solution interval for angle ***4x*** to degrees, namely **[0°, 720°)**, and then we continue to work with degrees.

Therefore, with the calculator in degree mode, we get ***4x* = 120°**.

Solve Simple Trigonometric Equations involving Multiples of an Angle (5 of 12)

Example 2 continued:

We found $\boldsymbol{4x = 120^{o}}$ in the previous step. We will now find its reference angle.

Here we must be careful. We did not get a QI angle. This is because of the way the calculator is programed. Remember, the range for the arccosine is between $\mathbf{0^{o}}$ and $\mathbf{180^{o}}$. **The reference angle is**

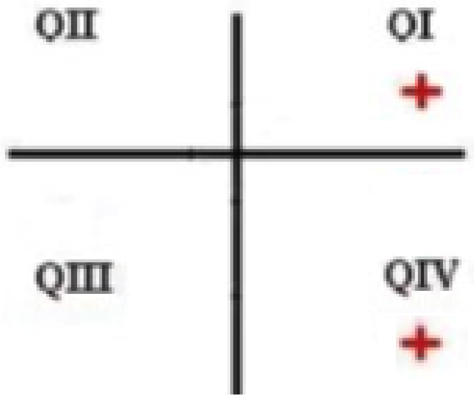
$\mathbf{180^{o} – 120^{o} = 60^{o}}$.

Solve Simple Trigonometric Equations involving Multiples of an Angle (6 of 12)

Example 2 continued:

We have a reference angle, and we know that the value of the cosine ratio is negative.

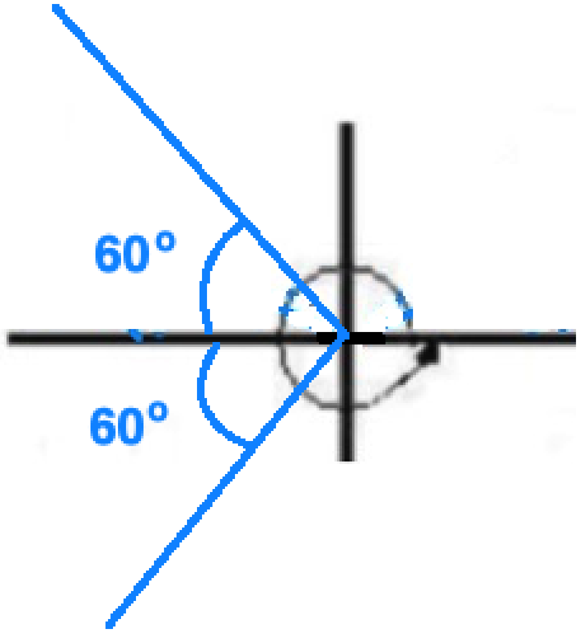
Therefore, we will use *All Students Take Calculus* to find the quadrants in which the cosine ratio is positive. Thus, the cosine is negative in QII and QIII. See picture below!



Solve Simple Trigonometric Equations involving Multiples of an Angle (7 of 12)

Example 2 continued:

Finally, let's first indicate the solution interval for angle ***4x***. Since it is **[0°, 720°)**, which is rather large, we only graph the angle **360°**. Then we draw the reference angle in the appropriate quadrants.



We can now find the solutions for angle ***4x*** in the interval **[0°, 360°)** with the help of the picture.

Solution in QII:

$\boldsymbol{4x_2 = 180^\circ - 60^\circ = 120^\circ}$

Solution in QIII:

$\boldsymbol{4x_2 = 180^\circ + 60^\circ = 240^\circ}$

Solve Simple Trigonometric Equations involving Multiples of an Angle (8 of 12)

Example 2 continued:

Because we are supposed to find solutions for angle ***4x*** on the interval **[0°, 720°)**, we must add another 360° rotation as follows:

$\boldsymbol{4x_3 = 30^o + 360^o = 390^o}$ $\qquad$ $\boldsymbol{4x_4 = 150^o + 360^o = 510^o}$

Since we are supposed to express the solutions in terms of radians, we simply change the degree solutions to radians to get the following:

$\boldsymbol{4x_1 = 30^o \equiv \frac{\pi}{6}}$ $\qquad$ $\boldsymbol{4x_2 = 150^o \equiv \frac{5\pi}{6}}$

$\boldsymbol{4x_3 = 390^o \equiv \frac{13\pi}{6}}$ $\qquad$ $\boldsymbol{4x_4 = 510^o \equiv \frac{17\pi}{6}}$

Solve Simple Trigonometric Equations involving Multiples of an Angle (9 of 12)

Step 6 - Find the solutions for angle ***x*** by solving the solutions of angles ***bx*** for ***x***.

Example 2 continued:

We were asked to solve for the angle ***x***, and so far we solved for angle ***4x***. Therefore, we must now divide both sides of the four equations on the previous slide by 4 to finally get the solutions for angle ***x***.

Given $4x_1 = \frac{\pi}{6}$, then $x_1 = \frac{\pi}{24}$

Given $4x_2 = \frac{5\pi}{6}$, then $x_2 = \frac{5\pi}{24}$

Given $4x_3 = \frac{13\pi}{6}$, then $x_3 = \frac{13\pi}{24}$

Given $4x_4 = \frac{17\pi}{6}$, then $x_4 = \frac{17\pi}{24}$

Solve Simple Trigonometric Equations involving Multiples of an Angle (10 of 12)

Example 3:

Solve $\boldsymbol{5\,cos\,\frac{1}{2}\,x = 0}$ for ***x*** in the interval **[0º, 720º)**. Express the solutions in EXACT degrees.

The solution interval for angle ***x*** is **[0º, 720º)**. Then the solution interval for angle $\boldsymbol{\frac{1}{2}\,x}$ is $\left[\boldsymbol{\frac{1}{2}\,(0^{o}), \frac{1}{2}\,(720^{o})}\right)$ or **[0º, 360º)**.

Next, we will isolate the trigonometric ratio by dividing both sides of the equal sign by 5.

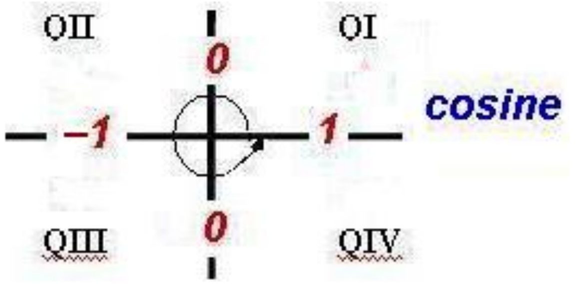
We end up with $\boldsymbol{cos\,\frac{1}{2}\,x = 0}$.

Solve Simple Trigonometric Equations involving Multiples of an Angle (11 of 12)

We are going to use the concept of inverse trigonometric functions to solve for the angle $\frac{1}{2}\boldsymbol{x}$ in the equation from Step 1 using the calculator.

Given $\boldsymbol{cos\,\frac{1}{2}x = 0}$, we find $\boldsymbol{\frac{1}{2}x = cos^{-1}}(0)$.

With the calculator in degree mode, we get $\boldsymbol{\frac{1}{2}x = 90^o}$ which is a quadrantal angle. It does not have a reference angle! Therefore, we will use the following picture:



Solve Simple Trigonometric Equations involving Multiples of an Angle (12 of 12)

From the picture on the previous slide, we find that $\boldsymbol{cos\,\frac{1}{2}x}$ equals **0** at **90°** and **270°** on the interval **[0, 360°)**.

Therefore, we have two solutions, namely $\boldsymbol{\frac{1}{2}x_1 = 90°}$ and $\boldsymbol{\frac{1}{2}x_2 = 270°}$.

BUT we are asked to solve for the angle $\boldsymbol{x}$ and so far we solved for angle $\boldsymbol{\frac{1}{2}x}$. Therefore, we must now divide both sides of the two equations on the previous slide by $\frac{1}{2}$ to finally get the solutions for angle $\boldsymbol{x}$.

Given $\boldsymbol{\frac{1}{2}x_1 = 90°}$, then $\boldsymbol{x_1 = 180°}$

Given $\boldsymbol{\frac{1}{2}x_2 = 270°}$, then $\boldsymbol{x_2 = 540°}$