Concepts
Matrices and Systems of Linear Equations

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Define a matrix (plural: matrices).
2. Perform allowable matrix row operations.
3. Define augmented matrices.
4. Solve systems of linear equations using the matrix method.

NOTE: This lesson contains some examples. You can find more examples in the “Examples” document also located in the appropriate MOM Learning Materials folder.

1. Definition of a Matrix (1 of 6)

In this lesson, we will learn another method for solving systems of linear equations. It relies heavily on something called a **matrix (plural: matrices**).

A matrix is a set of numbers, arranged in rows and columns and **placed in brackets.**

Sometimes, we give matrices names usually using capital letters, such as *A*, *B*, or *C*. Each number in a matrix is called an **element**.

For example, $C = \begin{bmatrix} 5 & -2 & 8 \\ -3 & 4 & 1 \\ 1 & 6 & 0 \end{bmatrix}$

Definition of a Matrix (2 of 6)

Every matrix is said to be of a certain **ORDER** which is usually expressed as ***m* × *n*** (*m by n*). The ***m*** indicates the number of rows, and the ***n*** indicates the number of columns. The row number is always stated first.

For example, $A = \begin{bmatrix} 5 & -2 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$ 3 × 2 matrix

or $B = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$ 2 × 2 matrix, also called a square matrix because the number of rows equal the number of columns.

Definition of a Matrix (3 of 6)

Addition and Subtraction of Matrices:

Matrices must have the same number of rows and columns to perform addition and subtraction. Each element in the corresponding position is then added or subtracted.

Example 1:

Perform the indicated matrix operation: $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

Definition of a Matrix (4 of 6)

Example 2:

Perform the indicated matrix operation: $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

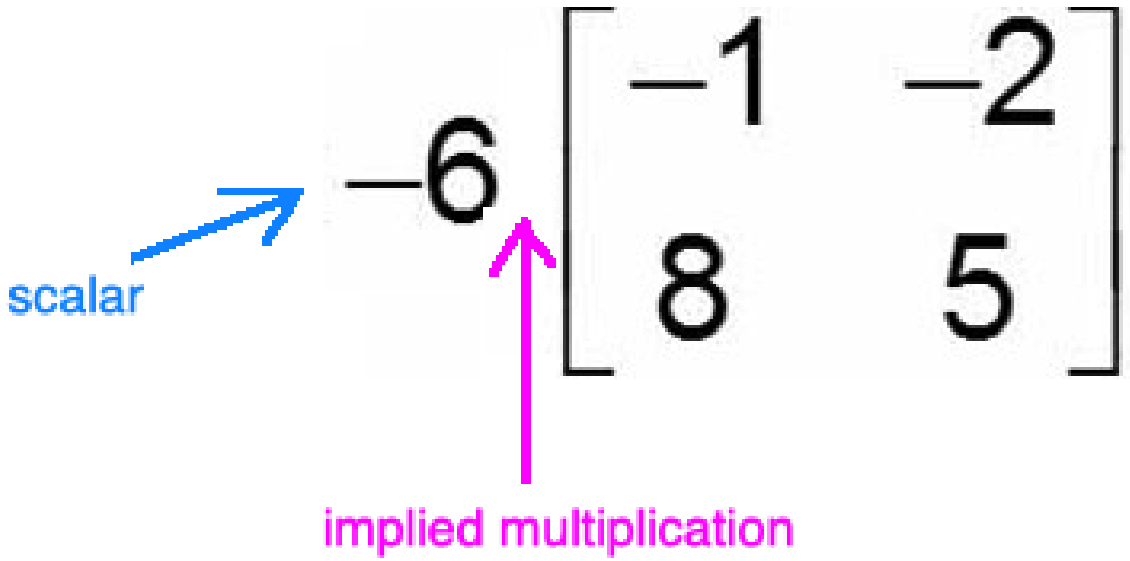
Solution:

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -4-6 & 3-(-3) \\ 7-2 & -6-(-4) \end{bmatrix} = \begin{bmatrix} -10 & 6 \\ 5 & -2 \end{bmatrix}$$

Definition of a Matrix (5 of 6)

Multiplication of Matrices:

There are several different types of matrix multiplications. The only one we will discuss is called **scalar multiplication**. A **scalar** in matrix mathematics is a single real number.



Definition of a Matrix (6 of 6)

Example 3:

If $B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$, find the matrix $-6B$. Note that -6 is a **scalar**.

Solution:

$$-6B = -6\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} (-6)(-1) & (-6)(-2) \\ (-6)(8) & (-6)(5) \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$$

2. Matrix Row Operations (1 of 6)

Any row in a matrix can be transformed by performing one or more of the following three row operations:

1. All elements in any row R of a matrix may be multiplied by some nonzero scalar, let's call it k. Symbolically, this can be expressed as kR_i where i indicates a certain row in the matrix.

   Example 4:

   Given the matrix $\begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix}$, replace Row 1 by carrying out $\frac{1}{4}R_1$.

   This means we multiply all elements in Row 1 by the scalar $\frac{1}{4}$.

Matrix Row Operations (2 of 6)

Example 4 continued:

We write the instructions $\frac{1}{4}R_1$ next to Row 1. Let's immediately copy Rows 2 and 3 since we are not making changes to them.

$$\begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix} \begin{matrix} \frac{1}{4}R_1 \\ \\ \\ \end{matrix} = \begin{bmatrix} \frac{1}{4}(4) & \frac{1}{4}(12) & \frac{1}{4}(-20) \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix}$$

Matrix Row Operations (3 of 6)

2. Two rows *R* in a matrix can be added, and their sum can replace any row. Symbolically, this can be expressed as $kR_i + R_j$ where *i* and *j* indicate certain rows in the matrix. The letter *k* is some nonzero scalar.

Example 5:

Given the matrix $\begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix}$, replace Row 3 by carrying out $3R_2 + R_3$.

This means we multiply all elements in Row 2 by the scalar 3 and then add to this product the "corresponding" elements in Row 3.

Matrix Row Operations (4 of 6)

Example 5 continued:

We write the instructions $3R_2 + R_3$ next to Row 3. Let's immediately copy Rows 1 and 2 since we are not making changes to them.

$$\begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix} \begin{matrix} \\ \\ 3R_2 + R_3 \end{matrix} = \begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ 3(1)+(-3) & 3(6)+(-2) & 3(-3)+1 \end{bmatrix}$$

3R2 R3 3R2 R3 3R2 R3

$$= \begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ 0 & 16 & -8 \end{bmatrix}$$

Matrix Row Operations (5 of 6)

3. Two rows *R* of a matrix may be interchanged. Symbolically, this can be expressed as $R_i \leftrightarrow R_j$ where *i* and *j* indicate certain rows in the matrix.

Example 6:

Given the matrix $\begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix}$, carry out $R_1 \leftrightarrow R_2$.

This means we want to interchange Row 1 and Row 2.

Matrix Row Operations (6 of 6)

Example 6 continued:

We write the instructions $R_1 \leftrightarrow R_2$ in between Rows 1 and 2. Let's immediately copy Row 3 since we are not making changes to it.

$$\begin{bmatrix} 4 & 12 & -20 \\ 1 & 6 & -3 \\ -3 & -2 & 1 \end{bmatrix} R_1 \leftrightarrow R_2 = \begin{bmatrix} 1 & 6 & -3 \\ 4 & 12 & -20 \\ -3 & -2 & 1 \end{bmatrix}$$

3. Definition of Augmented Matrices (1 of 2)

We will now add a vertical bar to separate the rightmost column of the matrix from the other columns. We call this an augmented matrix.

For example, $\begin{bmatrix} 4 & 12 & 8 \\ -3 & 6 & 7 \end{bmatrix}$ becomes $\left[\begin{array}{cc|c} 4 & 12 & 8 \\ -3 & 6 & 7 \end{array}\right]$.

or

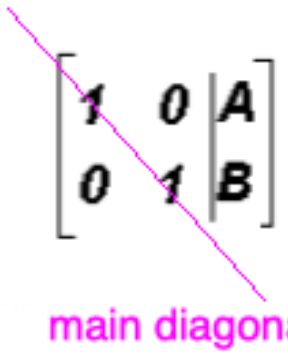
$\begin{bmatrix} 4 & 12 & -20 & 8 \\ 7 & 6 & -3 & 7 \\ -3 & -2 & 4 & -9 \end{bmatrix}$ becomes $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 7 & 6 & -3 & 7 \\ -3 & -2 & 4 & -9 \end{array}\right]$.

Definition of Augmented Matrixes (2 of 2)

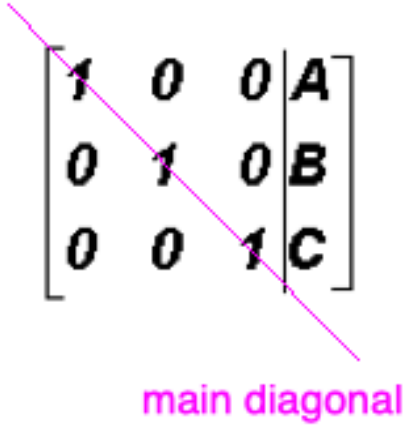
Row-Echelon Form of an Augmented Matrix

An augmented matrix is said to be in row-echelon form if the following occurs on the left side of the vertical bar:

- The main diagonal contains only elements of 1. Note, the main diagonal runs from the top left corner element to the bottom right corner element both are left of the vertical bar.
- The elements 0 are above and below the elements 1.



A and *B* are any real numbers.



A, *B*, and *C are* any real numbers.

4. Solve Systems of Linear Equations Using the Matrix Method (1 of 9)

In earlier lectures, we solved systems of linear equations using the Substitution and/or Elimination Methods. Now we are going to the *Matrix Method* to solve systems of linear equations.

This method is also called the *Gauss-Jordan Elimination Method*. It was created by the German mathematician Carl Friedrich Gauss (1777-1855) and the German geodesist (someone who measures and monitors the earth) Wilhelm Jordan (1842-1899).

Solve Systems of Linear Equations Using the Matrix Method
(2 of 9)

Strategy for Solving Systems of Linear Equations Using the Matrix Method:

Step 1 - Place the equations in a system into an augmented matrix using only the coefficients of the variables. Coefficients of like variables appear in the same column.

NOTE: In mathematics, we use alphabetic order to write columns. For example, the *x*-column is first, then the *y*-column, then the *z*-column, and finally the constant column.

Solve Systems of Linear Equations Using the Matrix Method
(3 of 9)

Example 7:

Solve the, the system $\begin{cases} 2x+y=1 \\ 3x+2y=4 \end{cases}$ using the Matrix Method.

Let's write the system into an augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right]$$

Note that the first column is the *x*-column and the second column the *y*-column. The vertical bar is the equal sign, and the last column consist of the constants.

Solve Systems of Linear Equations Using the Matrix Method
(4 of 9)

Step 2 - Change the augmented matrix from Step 1 to row-echelon form **using the three allowable matrix row operations discussed earlier**. The row operations MUST be completed in the order as indicated by the subscripts on the elements below!

$$\left[\begin{array}{cc|c} 1_1 & 0_4 & A \\ 0_2 & 1_3 & B \end{array}\right]$$

for two linear equations in two variables

$$\left[\begin{array}{ccc|c} 1_1 & 0_9 & 0_8 & A \\ 0_2 & 1_4 & 0_7 & B \\ 0_3 & 0_5 & 1_6 & C \end{array}\right]$$

for three linear equations in three variables

Solve Systems of Linear Equations Using the Matrix Method

(5 of 9)

Example 7 continued with $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right]$:

First, we want the number 1 for the first element in Row 1.

NOTE: It is easy to achieve 1's because all we have to do is divide all elements of a row by the appropriate number.

We first copy Row 2. Then we replace each element in Row 1 with the following calculation: $\frac{1}{2}R_1$

We write the instructions next to the row we want to change.

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array} = \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 3 & 2 & 4 \end{array}\right]$$

Solve Systems of Linear Equations Using the Matrix Method

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Example 7 continued with $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 3 & 2 & 4 \end{array}\right]$:

Next, we want the number 0 for the first element in Row 2.

We first copy Row 1. Then we replace each element in Row 2 with the following calculation: $-3R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 3 & 2 & 4 \end{array}\right] \begin{array}{l} \\ -3R_1 + R_2 \end{array} = \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \end{array}\right]$$

Note: We calculated the terms of the new Row 2 as follows:

1st term: $-3(1) + 3 = 0$ **2nd term:** $-3\left(\frac{1}{2}\right) + 2 = \frac{1}{2}$ **constant:** $-3\left(\frac{1}{2}\right) + 4 = \frac{5}{2}$

Solve Systems of Linear Equations Using the Matrix Method
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Example 7 continued with $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \end{array}\right]$:

Working with the changed matrix, we now want the number 1 for the second element in Row 2.

We first copy Row 1. Then we replace each element in Row 2 with the following calculation: $2R_2$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \end{array}\right] 2R_2 = \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 5 \end{array}\right]$$

Solve Systems of Linear Equations Using the Matrix Method

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Example 7 continued with $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 5 \end{array}\right]$:

We now want the number 0 for the second element in Row 1.

We first copy Row 2. Then we replace each element in Row 1 with the following calculation: $-\frac{1}{2}R_2 + R_1$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 5 \end{array}\right] \begin{array}{l} \\ -\frac{1}{2}R_2 + R_1 \end{array} = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

The matrix is now in row-echelon form and is therefore the **solution matrix**.

Note: We calculated the terms of the new Row 1 as follows:

1st term: $-\frac{1}{2}(0) + 1 = 1$ **2nd term:** $-\frac{1}{2}(1) + \frac{1}{2} = 0$ **constant:** $-\frac{1}{2}(5) + \frac{1}{2} = -2$

Solve Systems of Linear Equations Using the Matrix Method
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Step 3 - Once the augmented matrix is in row-echelon form it becomes the **solution matrix**. It allows us to identify the solution of the system by looking at the 1's!

Example 7 continued:

We found the **solution matrix** for the system $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right]$ to be $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$. It translates into $\mathbf{1}x + \mathbf{0}y = -\mathbf{2}$ and $\mathbf{0}x + \mathbf{1}y = \mathbf{5}$.

Simplified, this means $x = -2$ and $y = 5$. The solution to the given system is the ordered pair $(-2, 5)$. This is the point at which the graphs of the two linear equations in the system intersect.