Concepts and Examples
Introduction to Sequences and Series

Based on power point presentations by Pearson Education, Inc.

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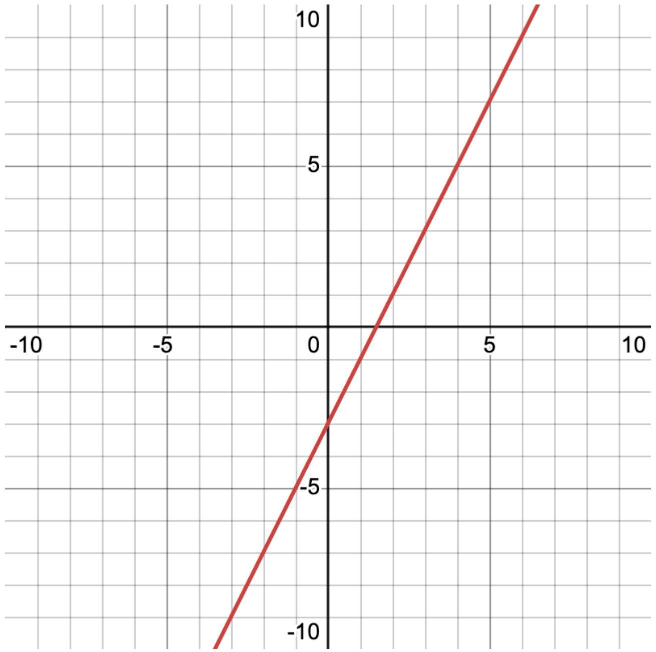
Learning Objectives

1. Memorize the definition of a mathematical sequence.
2. Use summation notation.
3. Memorize the definition of a mathematical series.

1. Definition a of a Sequence (1 of 4)

So far, we discussed functions whose domain consisted of an entire interval of numbers including integers, rational numbers, and irrational number. Therefore, we are usually able to visualize graphs as continuous segments.

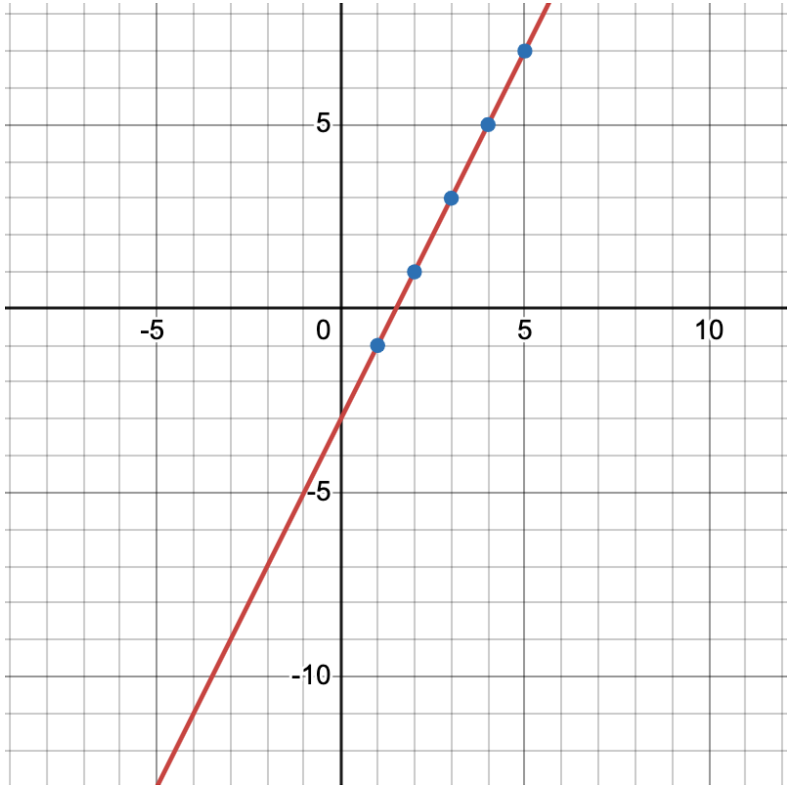
For example, below is the graph of the linear function $\boldsymbol{g(x) = 2x - 3}$, where we used all real numbers in the domain.



Definition a of a Sequence (2 of 4)

We will now discuss special functions called “sequences.” Their domain consists only of positive integers.

For example, in the linear function from the previous slide, the graph of the <u>related sequence</u> shown below would only consist of the (blue) points which are created by using a domain consisting of positive integers.



When discussing sequences, we will not use function notation such as $\boldsymbol{g(x)}$. Instead, we use $\boldsymbol{a_n}$.

For example, the sequence related to $\boldsymbol{g(x) = 2x - 3}$

is $\boldsymbol{a_n = 2n - 3}$ where the domain consists of $n = 1, 2, 3, 4, 5$ and so on.

Definition a of a Sequence (3 of 4)

In summary, a sequence $\boldsymbol{a_n}$ is a function whose domain consists of positive integers. We often call $\boldsymbol{a_n}$ the **general term**.

The values of the dependent variable, usually called terms, are represented by the first term called $\boldsymbol{a_1}$ where $n = 1$; the second term called $\boldsymbol{a_2}$ where $n = 2$; the third term called $\boldsymbol{a_3}$ where $n = 3$; and so.

For example, given the sequence $\boldsymbol{a_n = 2n - 3}$, the first term $\boldsymbol{a_1}$ is -1, that's when $n = 1$.

Sequences whose domains consist of a finite number of positive integers are called **finite sequences**. They are often expressed as $a_1, a_2, a_3, a_4, \ldots, a_n$.

For example, the sequence $\boldsymbol{a_n = 2n - 3}$ with n from 1 to 5 can be expressed as $-1, 1, 3, 5, 7$.

Definition a of a Sequence

Sequences whose domain consists of infinitely many positive integers are called **infinite sequences**. They are often expressed as $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$.

For example, the sequence $\boldsymbol{a_n = 2n - 3}$ with n from 1 to infinity can be expressed as $-1, 1, 3, 5, 7, \ldots$.

2. Summation Notation (1 of 3)

In mathematics it is often important to find the sum of a certain number of specific algebraic expressions . The upper-case Greek letter sigma Σ is used to indicate this sum. This is symbolized as follows:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$$ where

a_k is a specific algebraic expression

k is the index of summation

k = 1 is the lower limit of summation

n is the upper limit of summation

NOTE: The index of summation does not have to be ***k***. It can be any other lower-case letter of the alphabet. Furthermore, in general the lower limit of summation can be any integer other than 1.

Summation Notation (2 of 3)

Example 1:

Expand and evaluate the sum $\sum_{k=1}^{6} 2k^2$

Given that $a_k = 2k^2$, we let $k = 1$ for the first term. For the second term, we let $k = 2$ and so on until we get to the sixth term where $k = 6$! Then we expand the sum and evaluate it.

$$\begin{aligned}
& \quad\;\; {\scriptstyle k=1} \quad\; {\scriptstyle k=2} \quad\; {\scriptstyle k=3} \quad\; {\scriptstyle k=4} \quad\; {\scriptstyle k=5} \quad\; {\scriptstyle k=6} \\
\sum_{k=1}^{6} 2k^2 &= 2\cdot 1^2 + 2\cdot 2^2 + 2\cdot 3^2 + 2\cdot 4^2 + 2\cdot 5^2 + 2\cdot 6^2 \\
&= 2\cdot 1 + 2\cdot 4 + 2\cdot 9 + 2\cdot 16 + 2\cdot 25 + 2\cdot 36 \\
&= 2 + 8 + 18 + 32 + 50 + 72 \\
&= 182
\end{aligned}$$

Summation Notation (3 of 3)

Example 2:

Expand and evaluate the sum $\sum_{k=1}^{5} 4$.

Given that $a_k = 4$, we let $k = 1$ for the first term. For the second term, we let $k = 2$ and so on until we get to the fifth term where $k = 5$! Then we evaluate the sum.

Please note that there is no *k* in the term next to sigma! Therefore, every term in the sum is 4.

$$\sum_{k=1}^{5} 4 = \underset{k=1}{4} + \underset{k=2}{4} + \underset{k=3}{4} + \underset{k=4}{4} + \underset{k=5}{4}$$

3. Definition of a Series (1 of 2)

The sum of the terms of a sequence is called a series.

Finite Sequence: $a_1, a_2, a_3, a_4, \ldots, a_n$ (has commas)

Finite Series: $a_1 + a_2 + a_3 + a_4 + \ldots + a_n$ (has plus signs)

Since a series involves addition, we can use summation notation to write the finite series above as follows:

$$\sum_{k=1}^{n} a_k$$

Definition of a Series (1 of 2)

Example 3:

– 1, 1, 3, 5, 7 is a finite sequence where every term is produced by the general term $\boldsymbol{a_n = 2n + 3}$ starting at $n = 1$ and then using subsequent integers in turn.

– 1 + 1 + 3 + 5 + 7 is a finite series. Its terms are also produced by $\boldsymbol{2n + 3}$.

We can express this series in *Summation Notation* as $\sum_{k=1}^{5} (2k + 3)$ with the lower and upper limit indicating that we want to add the first 5 terms.