Concepts and Examples
Introduction to Sequences and Series

Based on power point presentations by Pearson Education, Inc.

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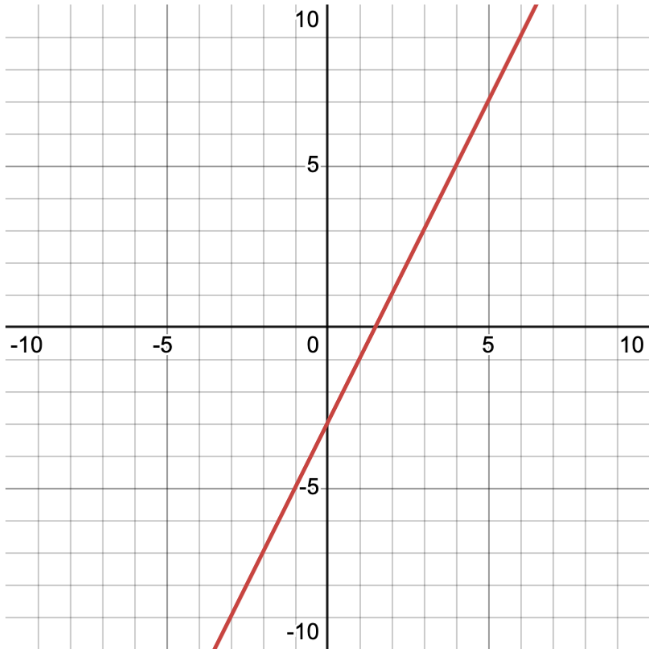
Learning Objectives

1. Memorize the definition of a mathematical sequence.
2. Use summation notation.
3. Memorize the definition of a mathematical series.

1. Definition a of a Sequence (1 of 4)

So far, we discussed functions whose domain consisted of real numbers including integers, rational numbers, and irrational number. Therefore, we are usually able to visualize graphs as continuous segments. Sequences, on the other hand, are functions whose domain strictly consists of positive integers.

Below is the graph of the linear function $\boldsymbol{g(x) = 2x - 3}$, where we used all real numbers in the domain. Therefore, it is a continuous line.



Definition a of a Sequence (4 of 4)

When discussing functions, we often use function notation such as $\boldsymbol{g(x)}$. When we discuss sequences, we replace function notation with a letter and subscript, for example, in this lesson we will use $\boldsymbol{a_n}$. We often call $\boldsymbol{a_n}$ the **general term**.

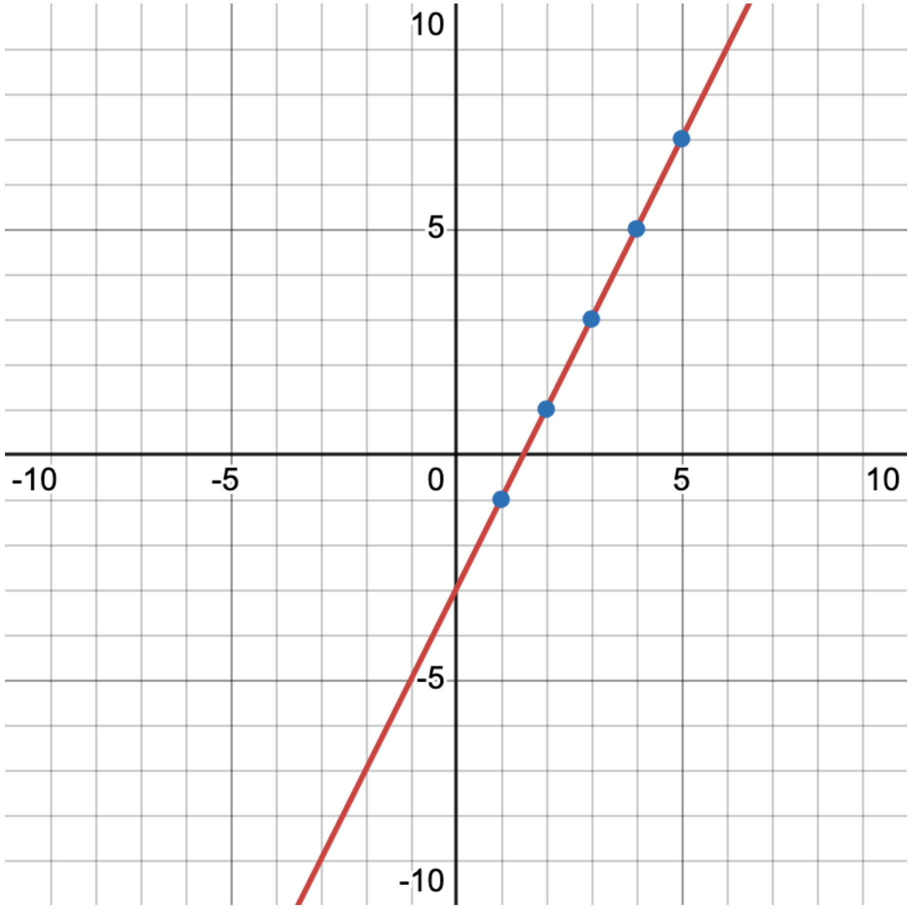
For example, the sequence related to $\boldsymbol{g(x) = 2x - 3}$ is $\boldsymbol{a_n = 2n - 3}$. Notice, we use $\boldsymbol{n}$ for the independent variable instead of $\boldsymbol{x}$.

In a sequence, the dependent variables, usually called terms, are represented by the first term called $\boldsymbol{a_1}$ when $n = 1$; the second term called $\boldsymbol{a_2}$ when $n = 2$; the third term called $\boldsymbol{a_3}$ when $n = 3$; and so.

For example, given the sequence $\boldsymbol{a_n = 2n - 3}$, the first term $\boldsymbol{a_1}$ is $2(1) - 3 = -1$ (dependent variable), that's when $n = 1$ (independent variable).

Definition a of a Sequence (4 of 4)

Below is the graph (blue) dots of the sequence $\boldsymbol{a_n = 2n - 3}$ where the domain consists of n = 1, 2, 3, 4, 5. It is superimposed onto the related linear function $\boldsymbol{g(x) = 2x - 3}$ (red) line.



Definition a of a Sequence (4 of 4)

Sequences whose domains consist of a finite number of positive integers are called **finite sequences**. They are often expressed as $a_1, a_2, a_3, a_4, \ldots, a_n$.

NOTE: The three dots are called an ellipsis, and it is used in mathematics to mean "and so forth”. It indicates the omission of terms that follow an obvious pattern as indicated by included terms.

For example, the sequence $\boldsymbol{a_n = 2n - 3}$ with n from 1 to 50 can be expressed as $-1, 1, 3, 5, 7, \ldots, 97$ where $n = 1, 2, 3, 4, 5, \ldots 50$.

Sequences whose domain consists of infinitely many positive integers are called **infinite sequences**. They are often expressed as $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$.

For example, the sequence $\boldsymbol{a_n = 2n - 3}$ with n from 1 to infinity can be expressed as $-1, 1, 3, 5, 7, \ldots, 97 \ldots$ where $n = 1, 2, 3, 4, 5, \ldots 50, \ldots$ (and so forth).

2. Summation Notation (1 of 3)

Before we discuss a mathematical series, it is important to introduce the upper-case Greek letter sigma Σ. It instructs us to find the sum of a certain number of terms derived from some mathematical expression. This is symbolized as follows:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$$ where $\boldsymbol{a_k}$ is some mathematical expression and

$\boldsymbol{a_1, a_2, a_3, a_4, \ldots, a_n}$ are terms where at k = 1, 2, 3, 4 … ending with a term where $k = n$

k is called the index of summation **

***k* = 1** is called the lower limit of summation

n is called the upper limit of summation

** NOTE: The index of summation does not have to be ***k***. It can be any other lower-case letter of the alphabet.

Summation Notation (2 of 3)

Example 1:

Expand and evaluate the sum $\sum_{k=1}^{6} 2k^2$

(Pronounced as ”the sum of $2k^2$ from $k = 1$ to $k = 6$).

Given the mathematical expression $a_k = 2k^2$, we let $k = 1$ for the first term (lower limit). For the second term, we let $k = 2$ and so on until we get to the sixth term where $k = n = 6$ (upper limit)! Then we expand the sum and evaluate it.

$k = 1$ $k = 2$ $k = 3$ $k = 4$ $k = 5$ $k = n = 6$

$$\sum_{k=1}^{6} 2k^2 = 2\cdot 1^2 + 2\cdot 2^2 + 2\cdot 3^2 + 2\cdot 4^2 + 2\cdot 5^2 + 2\cdot 6^2$$

$$= 2\cdot 1 + 2\cdot 4 + 2\cdot 9 + 2\cdot 16 + 2\cdot 25 + 2\cdot 36$$

$$= 2 + 8 + 18 + 32 + 50 + 72$$

$$= 182$$

Summation Notation (3 of 3)

Example 2:

Expand and evaluate the sum $\sum_{k=1}^{5} 4$.

(Pronounced as ”the sum of 4 from k = 1 to k = 5).

Please note that there is no k in the mathematical expression next to the sigma symbol! Therefore, every term in the sum is 4.

Given the mathematical expression $a_k = 4$, we let k = 1 for the first term (lower limit). For the second term, we let k = 2 and so on until we get to the fifth term where k = n = 5 (upper limit)! Then we evaluate the sum.

$$\sum_{k=1}^{5} 4 = \overset{k=1}{4} + \overset{k=2}{4} + \overset{k=3}{4} + \overset{k=4}{4} + \overset{k=n=5}{4}$$

$$= 20$$

3. Definition of a Series (1 of 3)

The sum of the terms of a sequence is called a series.

Finite Sequence: $a_1, a_2, a_3, a_4, \ldots , a_n$ (has commas)

Finite Series: $a_1 + a_2 + a_3 + a_4 + \ldots + a_n$ (has plus signs)

Since a series involves addition, we can use summation notation to write the finite series above as follows using k:

$$\sum_{k=1}^{n} a_k$$

Definition of a Series (2 of 3)

For example,

– 1, 1, 3, 5, 7 is a finite sequence where every term is produced by the general term $\boldsymbol{a_n = 2n - 3}$ starting at $n = 1$ and then using subsequent integers in turn with the last one being 5.

– 1 + 1 + 3 + 5 + 7 is a finite series. Its terms are also produced by $\boldsymbol{2n - 3}$.

Definition of a Series (3 of 3)

Example 3:

Express the series $-1 + 1 + 3 + 5 + 7$ in *Summation Notation* using k for the index of summation. Its terms are produced by $\boldsymbol{2k - 3}$.

The lower limit is $k = 1$ and the upper limit is $k = n = 5$.

$$\sum_{k=1}^{5} (2k - 3)$$ (Pronounced as "the sum of $2k - 3$ from $k = 1$ to $k = 5$).

Please note the variable in the mathematical expression next to the sigma symbol must match the variable used in the lower limit of the index of summation!