Concepts and Examples
Applications with Exponential Equations

Based on power point presentations by Pearson Education, Inc.
Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Use the exponential growth and decay formula.
2. Use the logistic growth formula.

In this lesson, we will use formulas to model some real-life applications using exponential functions and equations.

NOTE: The formulas in this lesson were derived using mathematics beyond the scope of this course. They do NOT have to be memorized, but it is expected that you are able to use them.

1. Exponential Growth and Decay (1 of 7)

The important concept of **exponential growth** is that the growth rate of a quantity is increasing at a greater and greater rate without any limitations. The best examples of exponential growth are seen in bacteria.

On the other hand, **exponential decay** refers to quantities decreasing at a greater and greater rate without any limitations. The best example of exponential decay is seen in radioactivity.

The mathematical model for exponential growth and decay is given by the following formula:

$\boldsymbol{P = P_0\, e^{kt}}$ where $\boldsymbol{e}$ is the irrational number approximately equal to 2.72 and

P_0 is the original amount or size of the growing/decaying entity at time $t = 0$

P is the amount at some time t

k is a constant representing the growth or decay rate (positive value in growth and negative value in decay)

t is time

Exponential Growth and Decay

Example 1:

In the year 2000, the population of Africa was 807 million and by 2011 it had grown to 1,052 million. Find the **unrestricted exponential growth function** that models the data. Use the exponential growth model $P = P_0 e^{kt}$ in which t is the number of years after 2000.

The first thing we will find is the growth constant k. Let's round k to three decimal places.

Exponential Growth and Decay (3 of 7)

Example 1 continued:

Here is what we are given:

Time $t = 11$, (from year 2000 to year 2011)

The original amount $P_0 = 807$ million (population in 2000)

The amount after 11 years $P = 1{,}052$ million (population in 2011)

We will now replace the appropriate variables in $P = P_0\, e^{kt}$ as follows:

$1052 = 807e^{11k}$ (Note that we are leaving off the "millions"!)

Next, we isolate the exponential expression in the equation.

$$e^{11k} = \frac{1052}{807}$$

Exponential Growth and Decay (4 of 7)

Example 1 continued:

Now, we will take the natural logarithm of both sides! Of course, we could have also used log base 10.

$$\ln e^{11k} = \ln\left(\frac{1052}{807}\right)$$

Next, we use the Power Rule.

$$11k \ln e = \ln\left(\frac{1052}{807}\right)$$

and $11k = \ln\left(\frac{1052}{807}\right)$ Note that **lne** = **log$_e$ e** = 1!

Finally, we find $k = \dfrac{\ln\left(\frac{1052}{807}\right)}{11}$.

Exponential Growth and Decay

Example 1 continued:

Using the calculator, we find that k is approximately equal to 0.024.

Subsequently, the function that models the unrestricted growth of the population of Africa using the given data and an exponential growth model is as follows:

$P = 807e^{0.024t}$ where P_0 is 807 (million), the population of Africa in the year 2000.

Exponential Growth and Decay

Example 2:

A certain radioactive material decays in such a way that the mass in kilograms remaining after t years is given by the model $m(t) = 120e^{-0.018t}$.

The general form of exponential decay is $P = P_0 e^{kt}$, where

P_0 is the original amount or size of the growing/decaying entity at time $t = 0$
P is the amount at some time t
k is a constant representing the decay rate (negative value)
t is time
e is the irrational number approximately equal to 2.72

a. What is the decay rate in the given model?

   $k = -0.018$ (note that k is negative because this is a decay model)

Exponential Growth and Decay (7 of 7)

Example 2 continued:

b. What is the initial mass when time $t = 0$?

We are looking for P_0, and in the given model $m(t) = 120e^{-0.018t}$ it is 120 kg.

c. How much mass remains after 50 years? Round to 2 decimal places.

We are looking for P at time $t = 50$. Given the model, we need to find $m(50)$.

$m(50) = 120e^{-0.018(50)}$

Using the calculator **and being sure to place – 0.018 × 50 into parentheses**, we find that $P = m(50)$ is approximately equal to 48.79 kg which is the mass that remains after 50 years.

Following are the TI-30X IIS steps:

120	2nd	LN	(-)	0.018	×	50	)	Enter

2. Logistic Growth (1 of 8)

Unlike exponential growth where the growth rate is constant and a quantity grows exponentially without restrictions, in logistic growth a quantity's *growth rate* eventually decreases. For example, this may be due to lack of food in a population or negative environmental factors. The mathematical model for restricted growth is given by the following formula:

$$A = \frac{c}{1 + ae^{-bt}}$$, where

A is some quantity at any given time

a , *b*, and *c* are constants with $c > 0$ and $b > 0$

t is time

e is the irrational number approximately equal to 2.72

Logistic Growth (2 of 8)

Example 3:

It was found that the logistic growth model $A = \dfrac{20\,000}{1+10e^{-1.5t}}$ illustrates the number ***A*** of people who contracted a viral flu ***t*** weeks after its initial outbreak in a town with 20,000 inhabitants.

a. How many people became ill with the flu when the epidemic began? Round to a whole number.

The time at the beginning of an event is always considered to be ***t = 0***.

$$A = \frac{20\,000}{1+10e^{-1.5(0)}} = \frac{20000}{1+10}$$

Using the calculator, we find that ***A*** is approximately equal to ***1818***.

The number of people who became ill with the flu when the epidemic began is 1818.

Logistic Growth

Example 3 continued:

b. How many people were ill by the end of the fourth week? Do not round until the final answer. Then round to a whole number.

Here ***t* = *4***. Then $A = \frac{20000}{1+10e^{-1.5(4)}}$.

Let's do a preliminary multiplication in the exponent as follows. This will make the calculator input easier.

$$A = \frac{20000}{1+10e^{-6}}$$

CALCULATOR TIP: We will input the ENTIRE expression into the TI-30X IIS calculator since we should never round until the very end.

Logistic Growth (4 of 8)

Example 3 continued:

Following is the calculator input:

20000	÷	(	1	+	10	^	(-)	6	)	Enter

We find that ***A*** is approximately equal to ***19516***.

In summary, 19,516 people became ill with the flu by the end of the fourth week.

Logistic Growth (5 of 8)

Example 3 continued:

c. How many weeks after outbreak will 10,000 people be infected with the flu? Round to one decimal place.

Since ***A*** illustrates the of people who contracted a viral flu, we let ***A = 10000*** and then we will find ***t***.

$$10000 = \frac{20000}{1+10e^{-1.5t}}$$

To solve this, we will first multiply both sides of the equality by the denominator.

$$10000\left(1+10e^{-1.5t}\right) = \frac{20000}{1+10e^{-1.5t}}\left(1+10e^{-1.5t}\right)$$

Logistic Growth (6 of 8)

Example 3 continued given $10000\left(1+10e^{-1.5t}\right)=\frac{20000}{1+10e^{-1.5t}}\left(1+10e^{-1.5t}\right)$:

We will continue to isolate ***t*** by using the *Distributive Property* as follows:

$$10000 + 100000e^{-1.5t} = 20000$$

Next, we will isolate the exponential expression as follows:

$$100000e^{-1.5t} = 10000$$

$$e^{-1.5t} = \frac{10000}{100000} = 0.1$$

Now, we can use logarithms to solve for ***t***.

Logistic Growth

Example 3 continued:

We continue as follows using the natural logarithm:

$ln\ e^{-1.5t} = ln\ 0.1$

Finally, we will use the *Power Rule*.

$-1.5\ t\ ln\ e = ln\ 0.1$

$-1.5\ t = ln\ 0.1$ Note that **lne** = $\mathbf{log_e\ e}$ = 1!

and $t = \dfrac{ln\ 0.1}{-1.5}$

Using the calculator, we find that *t* is approximately equal to 1.5.

Subsequently we find, that approximately 1.5 weeks after the outbreak of the flu, 10,000 people were infected.

Logistic Growth

Example 3 continued:

In case you wonder why $t = 1.5$ ended up being positive given a negative denominator ... any logarithm with an argument between 0 and 1 is negative! Here we are dividing a negative numerator by a negative denominator!