



Concepts and Examples

Applications with Exponential Equations

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Use the exponential growth and decay formula.
2. Use the logistic growth formula.
3. Use the compounded interest formulas.

In this lesson, we will use formulas to model some real-life applications using exponential functions and equations. An exponential model is a type of function that shows how a quantity grows exponentially over time. We often use exponential models to represent things like exponential growth and decay and compounded interest.

NOTE: The formulas in this lesson were derived using mathematics beyond the scope of this course. **They do NOT have to be memorized, but it is expected that you are able to use them.**

ATTENTION: We will use the calculator when working with the formulas in this lesson. However, first we will try to “cleaned up” the calculation by hand just a little. Once this is accomplished, we will input the ENTIRE formula into the calculator, any parentheses will be included. We DO NOT use intermediate rounding steps!

1. Exponential Growth and Decay (1 of 9)

The important concept of exponential growth is that the growth rate of a “population” is increasing at a greater and greater rate without any limitations. The best examples of exponential growth are seen in bacteria and compounded interest. On the other hand, exponential decay refers to quantities decreasing at a greater and greater rate without any limitations. The best example of exponential decay is seen in radioactivity.

The mathematical model for exponential growth and decay is given by the following formula:

$$P = P_0 e^{kt} \quad \text{where}$$

P_0 is the original amount or size of the growing entity at time $t = 0$

p is the amount at time t

k is a constant representing the growth or decay rate (positive value in growth and negative value in decay)

Exponential Growth and Decay (2 of 9)

Example 1:

In the year 2000, the population of Africa was 807 million and by 2011 it had grown to 1,052 million.

Find the **unrestricted exponential growth function** that models the data. Use the exponential growth model $P = P_0 e^{kt}$ in which t is the number of years after 2000.

We must first find the growth constant k which will then allow us to find the function that models the given data.

Exponential Growth and Decay (3 of 9)

Example 1 continued:

Here is what we are given:

Time $t = 11$, (from year 2000 to year 2011)

The original amount $P_0 = 807$ million (population in 2000)

The amount after 11 years $p = 1,052$ million (population in 2011)

We will now replace the appropriate variables in $P = P_0 e^{kt}$ as follows:

$$1052 = 807e^{11k} \quad (\text{Note that we are leaving off the "millions"!)}$$

Next, we isolate the exponential expression in the equation.

$$e^{11k} = \frac{1052}{807}$$

Exponential Growth and Decay (4 of 9)

Example 1 continued:

Now, we will take the natural logarithm of both sides! Of course, we could have also used log base 10.

$$\ln e^{11k} = \ln \left(\frac{1052}{807} \right)$$

Next, we use the Power Rule.

$$11k \ln e = \ln \left(\frac{1052}{807} \right)$$

$$\text{and } 11k = \ln \left(\frac{1052}{807} \right) \quad \text{Note that } \ln e = \log_e e = 1!$$

$$\text{Finally, we find } k = \frac{\ln \left(\frac{1052}{807} \right)}{11} .$$

Exponential Growth and Decay (5 of 9)

Example 1 continued:

Using the calculator, we find that k is approximately equal to 0.024.

Subsequently, the function that models the unrestricted growth of the population of Africa using the given data and an exponential growth model is $P = 807e^{0.024t}$.

Exponential Growth/Decay (6 of 9)

Example 2:

The half-life of strontium-90 is 28 years, meaning that after 28 years a given amount of the substance will have decayed to half the original amount.

Find the **exponential decay function** that models the data. Use the exponential decay model $P = P_0 e^{kt}$.

We must find the decay constant k which will then allow us to find the function that models the given data.

Here is what we are given:

Time $t = 28$

Some original amount P_0

What about P ? We know that after 28 years the amount of strontium-90 present is half the original amount. Thus, we can represent P as $\frac{P_0}{2}$.

Exponential Growth/Decay (7 of 9)

Example 2 continued:

We can now replace the appropriate variables in $P = P_0 e^{kt}$ as follows:

$$\frac{P_0}{2} = P_0 e^{28k}$$

Dividing both sides by P_0 , we get

$$\frac{1}{2} = e^{28k}$$

Now, we take the natural logarithm of both sides.

Exponential Growth/Decay (8 of 9)

Example 2 continued:

$$\ln\left(\frac{1}{2}\right) = \ln e^{28k}$$

$$\ln\left(\frac{1}{2}\right) = 28k \ln e$$

Note that $\ln e = \log_e e = 1!$

$$\ln\left(\frac{1}{2}\right) = 28k$$

We find $k = \frac{\ln\left(\frac{1}{2}\right)}{28}$.

Exponential Growth/Decay (9 of 9)

Example 2 continued:

Using the calculator, we find that k is approximately equal to -0.0248 .

Please note that k is negative because the numerator is negative. Any logarithm with an argument between 0 and 1 is negative!

The function that models the decay of strontium-90 using the given data and an exponential decay model is $P = P_0 e^{-0.0248t}$ where P_0 can be any amount.

2. Logistic Growth (1 of 7)

Unlike exponential growth where the growth rate is constant and the “population” grows exponentially without restrictions, in logistic growth a “population’s” *growth rate* (not the population itself) eventually decreases as its size approaches a maximum level. The maximum level is restricted by the environment’s finite supply of resources (i.e. the amount of food).

The mathematical model for restricted growth is given by the following formula:

$$A = \frac{c}{1 + ae^{-bt}}, \text{ where}$$

A is the number of “cases” at any given time
 a , b , and c are constants with $c > 0$ and $b > 0$
 e is the Euler number
 t is time

Logistic Growth (2 of 7)

Example 3:

It was found that the logistic growth model $A = \frac{20000}{1 + 10e^{-1.5t}}$ illustrates the number A of people who contracted a viral flu t weeks after its initial outbreak in a town with 20,000 inhabitants.

a. How many people became ill with the flu when the epidemic began? Round to a whole number.

The time at the beginning of an event is always considered to be $t = 0$.

$$A = \frac{20000}{1 + 10e^{-1.5(0)}} = \frac{20000}{1 + 10}$$

Using the calculator, we find that A is approximately equal to **1818**.

The number of people who became ill with the flu when the epidemic began is 1818.

Logistic Growth (3 of 7)

Example 3 continued:

b. How many people were ill by the end of the fourth week? Do not round until the final answer. Then round to a whole number.

Here $t = 4$. Then $A = \frac{20000}{1 + 10e^{-1.5(4)}} = \frac{20000}{1 + 10e^{-6}}$.

CALCULATOR TIP: We will input the ENTIRE expression into the calculator since we are not allowed to round until the very end.

We must do this with parentheses around the denominator to ensure that the calculator knows there are several terms. For the negative power, be sure to use the negative button (-) !

20000	÷	(1	+	10	^	(-)	6)	Enter
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Logistic Growth (4 of 7)

Example 3 continued:

We find that **A** is approximately equal to **19516**.

The number of people ill with the flu by the end of the fourth week is 19516.

- c. How many weeks after outbreak will 10,000 people be infected with the flu?
Round to one decimal place.

Since **A** illustrates the of people who contracted a viral flu, we let **A = 10000** and then we will find **t**.

$$10000 = \frac{20000}{1 + 10e^{-1.5t}}$$

To solve this, we will first multiply both sides of the equality by the denominator.

Logistic Growth (5 of 7)

Example 3 continued:

$$10000 \left(1 + 10e^{-1.5t}\right) = \frac{20000}{1 + 10e^{-1.5t}} \left(1 + 10e^{-1.5t}\right)$$

We will continue to isolate t by using the *Distributive Property* as follows:

$$10000 + 100000e^{-1.5t} = 20000$$

Next, we will isolate the exponential expression as follows:

$$100000e^{-1.5t} = 10000$$

$$e^{-1.5t} = \frac{10000}{100000} = 0.1$$

Now, we can use logarithms to solve for t .

Logistic Growth (6 of 7)

Example 3 continued:

We continue as follows:

$$\ln e^{-1.5t} = \ln 0.1$$

Now, we will use the *Power Rule*.

$$-1.5 t \ln e = \ln 0.1$$

$$-1.5 t = \ln 0.1$$

Note that **$\ln e = \log_e e = 1!$**

$$t = \frac{\ln 0.1}{-1.5}$$

Logistic Growth (7 of 7)

Example 3 continued:

Using the calculator, we find that t is approximately equal to 1.5.

In case you wonder why t is positive ... any logarithm with an argument between 0 and 1 is negative! Here we are dividing a negative numerator by a negative denominator!

Subsequently we find, that approximately 1.5 weeks after the outbreak of the flu, 10,000 people were infected.

3. Compounded Interest (1 of 12)

Compounded interest involves interest paid on an initial principal balance AND also on the amount of interest that accumulates. A bank may compound interest daily, weekly, monthly, quarterly, semi-annually, annually, and continuously.

To calculate weekly, monthly, quarterly, semi-annually, and annually compounded interest, we use the following formula:

$$A = P \left(1 + \frac{r}{n} \right)^{nt}, \text{ where}$$

P is the principal (the amount of money we want to save or borrow)

A is the accumulated amount

r is the annual interest rate (in decimal form)

t is the time the money is saved or borrowed

n is the number of times the interest is compounded per year

Compounded Interest (2 of 12)

To calculate continuously compounded interest, we use the following formula:

$$A = Pe^{rt}, \text{ where}$$

P is the principal (the amount of money we save or borrow)

A is the accumulated amount

e is the Euler number

r is the annual interest rate (in decimal form)

t is the number of years the money is saved or borrowed

NOTE: We will always use the ***e***-button (2nd + LN) on the calculator and not 2.72!

Compounded Interest (3 of 12)

Example 4:

A sum of \$20,000 is invested at an annual interest rate of 4%, compounded quarterly.

- Use the calculator to find the balance in the account after 6 years rounded to two decimal places.

Let's use $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where $P = 20000$, $r = 0.04$, $n = 4$ (quarterly), $t = 6$

$$A = 20000\left(1 + \frac{0.04}{4}\right)^{4(6)} = 20000\left(1 + \frac{0.04}{4}\right)^{24}$$

Note that we “cleaned up” the calculation by hand just a little. Now we will input it into a calculator in its ENTIRETY, parentheses and all. We DO NOT use intermediate rounding steps!

Compounded Interest (4 of 12)

Example 4 continued:

We find that **A** is approximately equal to \$25,394.69. **This is the balance in the account after 6 years.**

NOTE: Be sure to practice getting \$25,394.69 on your calculator! The input is as written above and includes the parentheses!

b. How much interest did we earn in 6 years?

All we have to do is subtract our initial investment from the balance in the account after 6 years.

That is, **$25394.69 - 20000 = 5394.69$**

We earned \$5,394.69 in interest in 6 years.

Compounded Interest (5 of 12)

Example 5:

A sum of \$3,500 is invested at an annual interest rate of 3.5%, compounded daily using the *Banker's Rule* (360 days in a year).

- Use the calculator to find the balance in the account after 3 years rounded to two decimal places.

Let's use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ where $P = 3500$, $r = 0.035$, $n = 360$ (daily), $t = 3$

$$A = 3500\left(1 + \frac{0.035}{360}\right)^{360(3)} = 3500\left(1 + \frac{0.035}{360}\right)^{1080}$$

Note that we “cleaned up” the calculation by hand just a little. Now we will input it into a calculator in its ENTIRETY, parentheses and all. We DO NOT use intermediate rounding steps!

Compounded Interest (6 of 12)

Example 5 continued:

We find that **A** is approximately equal to \$3,887.47. **This is the balance in the account after 3 years.**

NOTE: Be sure to practice getting \$3,887.47 on your calculator! The input is as written above and includes the parentheses!

b. How much interest did we earn in 3 years?

All we have to do is subtract our initial investment from the balance in the account after 3 years.

That is, **$3887.47 - 3500 = 387.47$**

We earned \$387.47 in interest in 3 years.

Compounded Interest (7 of 12)

Example 6:

Suppose we found a Certificate of Deposit (CD) that pays 4.3% interest compounded monthly for 5 years.

- a. If we deposit \$8,000 now, how much will you have in the account in 5 years? Round to the nearest cent!

Let's use $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where $P = 8000$, $r = 0.043$, $n = 12(\text{monthly})$, $t = 5$

$$A = 8000\left(1 + \frac{0.043}{12}\right)^{12(5)} = 8000\left(1 + \frac{0.043}{12}\right)^{60}$$

Note that we “cleaned up” the calculation by hand just a little. Now we will input it into a calculator in its ENTIRETY, parentheses and all. We DO NOT use intermediate rounding steps!

Compounded Interest (8 of 12)

Example 6 continued:

We find that **A** is approximately equal to \$9,915.08. **This is the balance in the account after 3 years.**

NOTE: Be sure to practice getting \$9,915.08 on your calculator! The input is as written above and includes the parentheses!

b. How much interest would we earn in 5 years?

All we have to do is subtract our initial investment from the balance in the account after 3 years.

That is, **$9915.08 - 8000 = 1915.08$**

We earned \$1,915.08 in interest in 3 years.

Compounded Interest (9 of 12)

Example 6 continued:

- c. Now suppose that we would like to have \$15,000 in the account in 3 years. How much would we need to deposit to get this? Round to the nearest cent.

Let's use $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where $A = 15000$, $r = 0.043$, $n = 12(\text{monthly})$, $t = 5$

$$15000 = P\left(1 + \frac{0.043}{12}\right)^{12(5)}$$

Note that we must solve for P which is the principal amount! Why don't we divide both sides by the following?

$$\left(1 + \frac{0.043}{12}\right)^{12(5)} = \left(1 + \frac{0.043}{12}\right)^{60}$$

Compounded Interest (10 of 12)

Example 6 continued:

We get
$$P = \frac{15000}{\left(1 + \frac{0.043}{12}\right)^{60}} .$$

Note that we will input the calculation into a calculator in its ENTIRETY, parentheses and all. We DO NOT use intermediate rounding steps!

We find that P is approximately equal to \$12,102.77169.

NOTE: Be sure to practice getting \$12,102.77169 on you calculator!
The input is as written above and includes the parentheses!

Since we are dealing with dollars and cents, we actually must put \$12,102.78 into the account. YES, we went against the rounding conventions and rounded UP. This is to ensure that we will indeed have \$15,000 at the end of 3 years.

Compounded Interest (11 of 12)

Example 7:

We want to invest \$10,000 and we are offered 5.85% interest per year compounded continuously. Use the calculator to find the balance in the account after 5 years rounded to two decimal places.

Let's use $A = Pe^{rt}$, where $P = 10000$, $r = 0.0585$, $t = 5$

$$A = 10000e^{0.0585(5)} = 10000e^{0.2925}$$

Note that we “cleaned up” the calculation by hand just a little. Now we will input it into a calculator in its ENTIRETY, parentheses and all. We DO NOT use intermediate rounding steps!

Compounded Interest (12 of 12)

Example 7 continued:

We will use the e -button on the calculator (2nd + LN) and NOT 2.72 as follows:

10000	2nd	LN	0.2925)	Enter
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Please note that we must insert the right parenthesis $)$ because when invoking e , a left parenthesis will open automatically.

We find that A is approximately equal to \$13,397.73. **This is the balance in the account after 5 years.**