Examples
Finding and Using Zeros of Polynomial Functions

Based on power point presentations by Pearson Education, Inc.

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Learning Objective

1. Use the *Zeros* of polynomial functions to better analyze their graphs.
2. Find the *Zeros* of factorable polynomial functions.

Example 1: Find *Zeros* of Polynomial Functions (1 of 3)

Find all *Zeros* of the polynomial function $f(x) = x^3 + 2x^2 - 4x - 8$ and state their multiplicity.

The degree of the function is 3, therefore, we must find 3 *Zeros*.

Step 1 - We find the *Zeros* of *f* by setting $f(x)$ equal to 0 and solving the resulting equation.

$$x^3 + 2x^2 - 4x - 8 = 0$$

Step 2 - Attempt factoring to write this polynomial equation as a product of factors.

Since the equation has four terms, we will attempt factoring by grouping.

Example 1: Find *Zeros* of Polynomial Functions (2 of 3)

We will start by grouping the first two factors and the last two factors as follows:

$(x^3 + 2x^2) + (-4x - 8) = 0$

Next, we will factor the greatest common factor out of every group. What we are hoping is to find a factor that both groups have in common.

The greatest common factor in the first group is x^2 and in the second group we will use – 4 instead of + 4 and you will see why in a moment.

$x^2(x + 2) - 4(x + 2) = 0$

Indeed, by factoring – 4 out of the second group, both resulting products have a factor of $(x + 2)$ in common. We will now factor it out.

$(x + 2)(x^2 - 4) = 0$

Example 1: Find *Zeros* of Polynomial Functions (3 of 3)

Step 3 - We can now attempt to use the *Zero Product Principle* to solve the equation for x.

We get $x + 2 = 0$ and $x^2 – 4 = 0$, and both equations can easily be solved.

Given $x + 2 = 0$, we determine $x = – 2$.

We find one real *Zero*, namely $x = – 2$.

Next, given $x^2 – 4 = 0$, we can use the *Difference of Squares* formula, the *Square Root Property,* or the *Quadratic Formula* to solve this.

Let's use the *Difference of Squares* formula to get $(x + 2)(x – 2) = 0$.

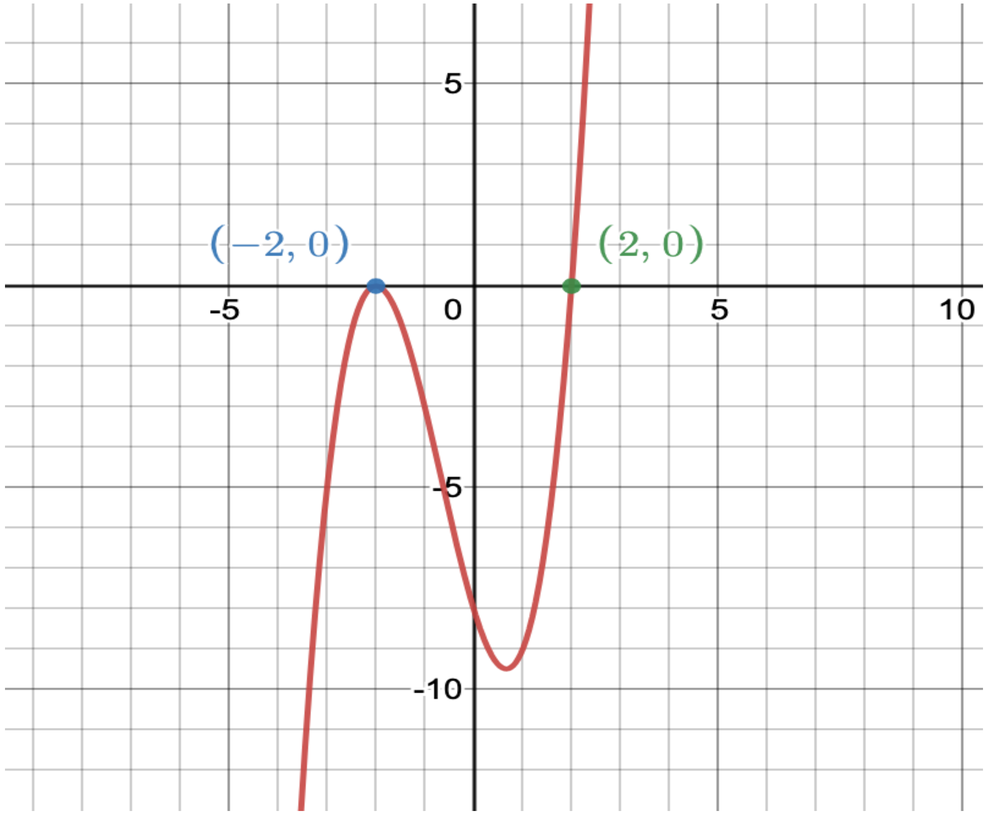
Then we use the *Zero Product Principle* to get $x + 2 = 0$ and $x – 2 = 0$.

We find two real *Zeros*, namely $x = – 2$ and $x = 2$.

In summary, the *Zeros* are – 2 (with multiplicity 2) and 2 (with multiplicity 1). The three *Zeros* are real numbers.

Example 2: Use *Zeros* of Polynomial Functions

Following is the graph of the polynomial function $f(x) = x^3 + 2x^2 - 4x - 8$ from Example 1. Note the end behavior given that the degree is odd and the leading coefficient is positive!



As you can see, the real *Zeros* – 2 and 2 are associated with the *x*-intercepts of the graph of the function.

Since – 2 is of EVEN multiplicity, the graph touches the *x*-axis mimicking a parabola.

Since 2 is of multiplicity 1, the graph crosses the *x*-axis in a straight line.

Example 3: Find *Zeros* of Polynomial Functions

Find all *Zeros* of the polynomial function $f(x) = x^5 – 4x^3$ and state their multiplicity.

The degree of the function is 5, therefore, we must find 5 *Zeros*.

Step 1 - We find the *Zeros* of *f* by setting *f*(*x*) *equal* to 0 and solving the resulting equation.

$$x^5 – 4x^3 = 0$$

Step 2 - Attempt factoring to write this polynomial equation as a product of factors.

We notice that both terms have the factor x^3 in common. We will factor it out.

$$x^3 (x^2 – 4) = 0$$

Example 3: Find *Zeros* of Polynomial Functions (2 of 3)

Step 3 - We can now attempt to use the *Zero Product Principle* to solve the equation for *x*.

We get $x^3 = 0$ and $x^2 - 4 = 0$, and both equations can easily be solved.

Given $x^3 = 0$, we will simply take the cube root of both sides of the equal sign to solve for *x*.

Please note that taking roots of ODD index of both sides of the equal sign does not require a $\pm$ in the solution.

We get $x = \sqrt[3]{0} = 0$. Note that $\sqrt[3]{x^3} = x$!

However, we must actually state that there are three real *Zeros*, namely $x = 0$, $x = 0$, and $x = 0$. Note that the *Zero* 0 is not distinct!

Example 3: Find *Zeros* of Polynomial Functions (3 of 3)

Next, given $x^2 - 4 = 0$, we can use the *Difference of Squares* formula, the *Square Root Property,* or the *Quadratic Formula* to solve this.

Let's use the *Square Root Property.* We must first write $x^2 = 4$.

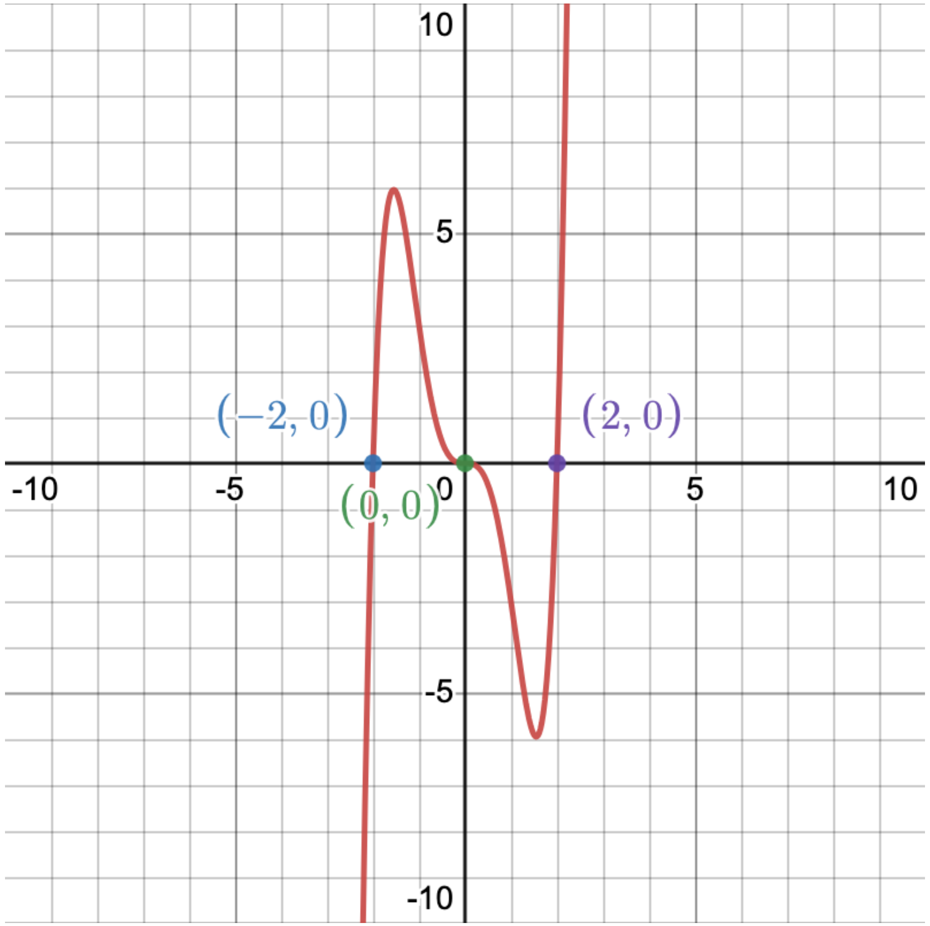
Then we can find that $x = \pm\sqrt{4} = \pm 2$.

We find two real *Zeros*, namely $x = -2$ and $x = 2$.

In summary, the *Zeros* are – 2 (with multiplicity 1), 2 (with multiplicity 1), and 0 (with multiplicity 3). The five *Zeros* are real numbers.

Example 4: Use *Zeros* of Polynomial Functions

Following is the graph of the polynomial function $f(x) = x^5 - 4x^3$ from Example 3. Note the end behavior given that the degree is odd and the leading coefficient is positive!



As you can see, the real *Zeros* – 2, 0, and 2 are associated with the *x*-intercepts of the graph of the function.

Since 0 is of ODD multiplicity and greater than 1, the graph crosses the *x*-axis mimicking a cubic function.

Since – 2 and 2 are of multiplicity 1, the graph crosses the *x*-axis in a straight line.

Example 5: Find *Zeros* of Polynomial Functions (1 of 3)

Find the real *Zeros* of $g(x) = 2(x + 3)(x - 7)^9$. Give their multiplicity and state if the graph crosses the *x*-axis or touches the *x*-axis at each *Zero*.

This function is written as a product of factors. Specifically, the factors are of the form $\boldsymbol{(x - r)^m}$. When this is the case, we can find the degree of the function by adding the exponents. That is, 1 + 9 = 10. It tells us that we must find 10 *Zeros*.

Step 1 - We find the *Zeros* of *f* by setting $g(x)$ *equal* to 0 and solving the resulting equation.

$$2(x + 3)(x - 7)^9 = 0$$

Step 2 - The function is already written as a product of factors.

Example 5: Find *Zeros* of Polynomial Functions

Step 3 - We can now attempt to use the *Zero Product Principle* to solve the equation for x.

We get $x + 3 = 0$ and $(x - 7)^9 = 0$, and both equations can easily be solved.

Given $x + 3 = 0$, we determine $x = -3$.

We find one real *Zero*, namely $x = -3$.

Given $(x - 7)^9 = 0$, we will simply take the ninth root of both sides of the equal sign to solve for x.

Please note that taking roots of ODD index of both sides of the equal sign does not require a $\pm$ in the solution.

Example 5: Find *Zeros* of Polynomial Functions (3 of 3)

We get $x - 7 = \sqrt[9]{0} = 0$. Note that $\sqrt[9]{(x-7)^9} = x - 7$!

Then $x = 0 + 7$ and $x = 7$.

However, we must actually state that 7 is a *Zero* of multiplicity 9.

In summary, the *Zeros* are – 3 (with multiplicity 1) and 7 (with multiplicity 9). The ten *Zeros* are real numbers.

Example 6: Use *Zeros* of Polynomial Functions

The graph of the polynomial function $g(x) = 2(x + 3)(x - 7)^9$ is impossible to show in a reasonable coordinate system!

Since – 3 is of multiplicity 1, the graph of the function crosses the *x*-axis at (– 3, 0) in a straight line.

On the other hand, 7 is of multiplicity 9 which is odd and indicates that the graph of the function crosses the *x*-axis at (7, 0) mimicking the picture of a cubic function .

Example 7: Find *Zeros* of Polynomial Functions (1 of 3)

Find all *Zeros* of the polynomial function $h(x) = x^4 - 8x^2 - 9$ and state their multiplicity.

The degree of the function is 4, therefore, we must find 4 *Zeros*.

Step 1 - We find the *Zeros* of *f* by setting *f*(*x*) equal to 0 and solving the resulting equation.

$$x^4 - 8x^2 - 9 = 0$$

Step 2 - Attempt factoring to write this polynomial equation as a product of factors.

We notice that the equation is quadratic in form. That is, its higher power is exactly twice the lower power. Therefore, we can try to use the short-cut factoring technique for the expressions $ax^2 + bx + c$ we learned in lesson 10PRE.

Example 7: Find *Zeros* of Polynomial Functions (2 of 3)

Finding all pairs of integers whose product is – 9, we notice that the pair – 9 and + 1 has a sum of – 9 + (+ 1) = – 8. Therefore, we can write $x^4 - 8x^2 - 9 = 0$ as follows:

$(x^2 - 9)(x^2 + 1) = 0$ Note that we used x^2 in the factors and not x!

Step 3 - We can now attempt to use *Zero Product Principle* to solve the equation for x.

We get $x^2 - 9 = 0$ and $x^2 + 1 = 0$, and both equations can easily be solved.

Given $x^2 - 9 = 0$, we can use the *Difference of Squares* formula, the *Square Root Property*, or the *Quadratic Formula* to solve this.

Let's use the *Difference of Squares formula* to get $(x + 3)(x - 3) = 0$.

Then we use the *Zero Product Principle* to get $x + 3 = 0$ and $x - 3 = 0$.

We find two real *Zeros*, namely $x = -3$ and $x = 3$.

Example 7: Find *Zeros* of Polynomial Functions (3 of 3)

Next, given $x^2 + 1 = 0$, we will use the *Square Root Prop*erty to solve for x. Please note that here we have a "sum of squares" for which there is no special formula. Incidentally, we could also use the *Quadratic Formula* to solve for x.

$x^2 = -1$

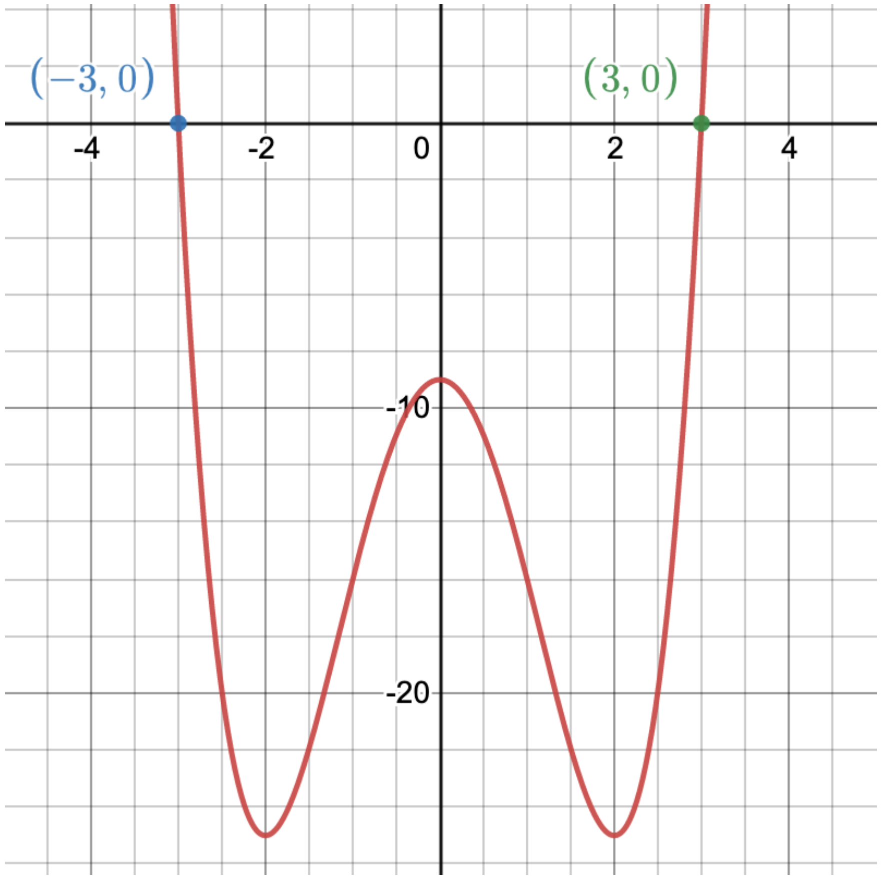
then $x = \pm\sqrt{-1} = \pm\, i$

We find two imaginary *Zeros*, namely $x = i$ and $x = -i$.

In summary, the *Zeros* are – 3 (with multiplicity 1), 3 (with multiplicity 1), *i* (with multiplicity 1, and – *i* (with multiplicity 1).

Example 8: Use *Zeros* of Polynomial Functions

Following is the graph of the polynomial function $h(x) = x^4 - 8x^2 - 9$ from Example 6. Note the end behavior given that the degree is even and the leading coefficient is positive!



As you can see, the real *Zeros* – 3 and 3 are associated with the *x*-intercepts of the graph of the function.

Since – 3 and 3 are of multiplicity 1, the graph crosses the *x*-axis in a straight line.

The imaginary *Zeros* CANNOT be seen on the graph, but they help shape it.

Example 9: Find *Zeros* of Polynomial Functions (1 of 3)

Find all *Zeros* of the polynomial function $f(x) = -x^4 + 16x^2$ and state their multiplicity.

The degree of the function is 4, therefore, we must find 4 *Zeros*.

Step 1 - We find the *Zeros* of *f* by setting $f(x)$ *equal* to 0 and solving the resulting equation.

$$-x^4 + 16x^2 = 0$$

Step 2 - Attempt factoring to write this polynomial equation as a product of factors.

We notice that both terms have the factor x^2 in common. We will actually factor out $-x^2$ and you will see why in a moment.

$-x^2(x^2 - 16) = 0$. Notice that one of the factors is is a *Difference of Squares*!

Example 9: Find *Zeros* of Polynomial Functions (2 of 3)

Step 3 - We can now attempt to use the *Zero Product Principle to solve for x*

We get $-x^2 = 0$ and $x^2 - 16 = 0$, and both equations can easily be solved.

Given $-x^2 = 0$, we will use the *Square Root Prop*erty to solve for *x*.

First, we will multiply both sides of the equal sign by – 1 to get

$x^2 = 0$ (There is no negative 0!)

Finally, $x = \pm\sqrt{0} = \pm 0$.

Despite the fact that there is no negative 0, we must state that there are two real *Zeros*, namely $x = 0$ and $x = 0$. Note that the *Zero* 0 is not distinct!

Example 9: Find *Zeros* of Polynomial Functions (3 of 3)

Next, given $x^2 - 16 = 0$, we can use the *Difference of Squares* formula, the *Square Root Property*, or the *Quadratic Formula* to solve this.

Let's use the *Difference of Squares* formula to get $(x + 4)(x - 4) = 0$.

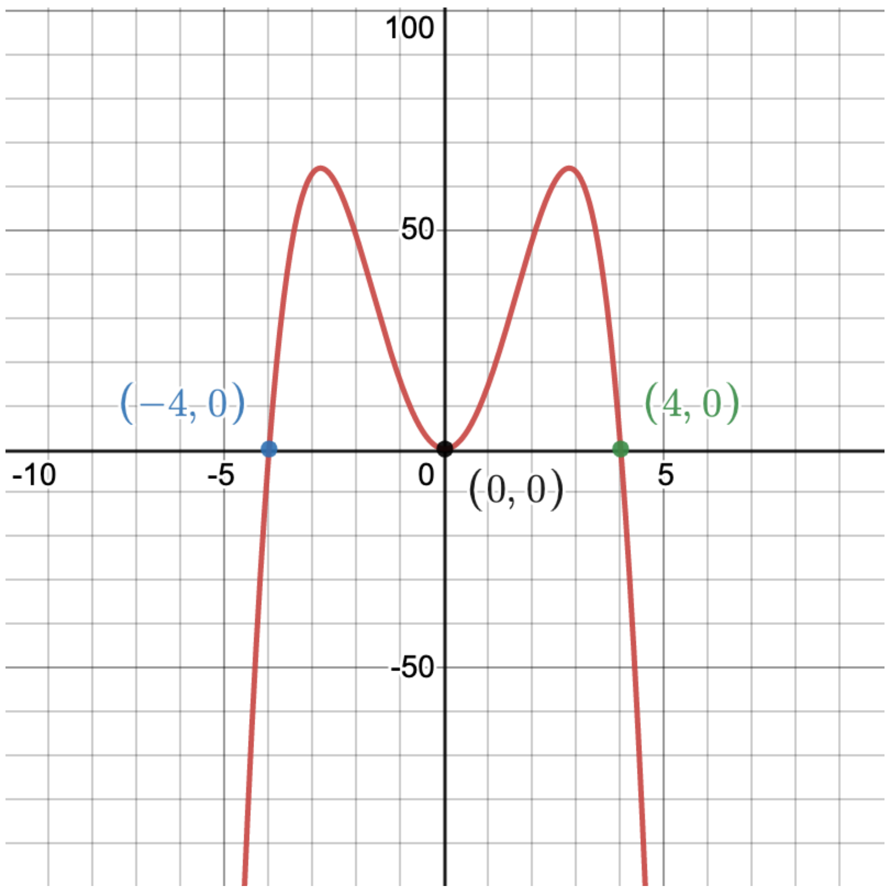
Then we use the *Zero Product Principle* to get $x + 4 = 0$ and $x - 4 = 0$.

We find two real *Zeros*, namely $x = -4$ and $x = 4$.

In summary, the *Zeros* are – 4 (with multiplicity 1), 4 (with multiplicity 1), and 0 (with multiplicity 2). The four *Zeros* are real numbers.

Example 10: Use *Zeros* of Polynomial Functions

Following is the graph of the polynomial function $f(x) = -x^4 + 16x$ from Example 8. Note the end behavior given that the degree is even and the leading coefficient is negative!



As you can see, the real *Zeros* – 4, 0, and 4 are associated with the *x*-intercepts of the graph of the function.

Since 0 is of EVEN multiplicity, the graph touches the *x*-axis mimicking a parabola.

Since – 4 and 4 are of multiplicity 1, the graph crosses the *x*-axis in a straight line.

Example 11: Find *Zeros* of Polynomial Functions (1 of 2)

Find all *Zeros* of the polynomial function $f(x) = x^4 - 2x^2$ and state their multiplicity.

The degree of the function is 4, therefore, we must find 4 *Zeros*.

Step 1 - We find the *Zeros* of *f* by setting $f(x)$ *equal* to 0 and solving the resulting equation.

$$x^4 - 2x^2 = 0$$

Step 2 - Attempt factoring to write this polynomial equation as a product of factors.

We notice that both terms have the factor x^2 in common. We will factor it out.

$$x^2 (x^2 - 2) = 0$$

Example 11: Find *Zeros* of Polynomial Functions (2 of 3)

Step 3 - We can now attempt to use the *Zero Product Principle* to solve the equation for *x*.

We get $x^2 = 0$ and $x^2 - 2 = 0$, and both equations can easily be solved.

Given $x^2 = 0$, we will use the *Square Root Property* to solve for *x*.

We get $x = \pm\sqrt{0} = \pm\, 0$.

Despite the fact that there is no negative 0, we must state that there are two real *Zeros*, namely, $x = 0$ and $x = 0$. Note that the *Zero* 0 is not distinct!

Example 11: Find *Zeros* of Polynomial Functions (3 of 3)

Next, given $x^2 - 2 = 0$, we will use the *Square Root Prop*erty to solve for x. Could we have used the *Difference of Squares* formula? Yes, since we can write 2 as $\sqrt{2} \cdot \sqrt{2}$ to get $(x - \sqrt{2})(x + \sqrt{2}) = 0$! We could have also used the *Quadratic Formula*.

Nevertheless, let's write $x^2 = 2$ and apply the *Square Root Property*.

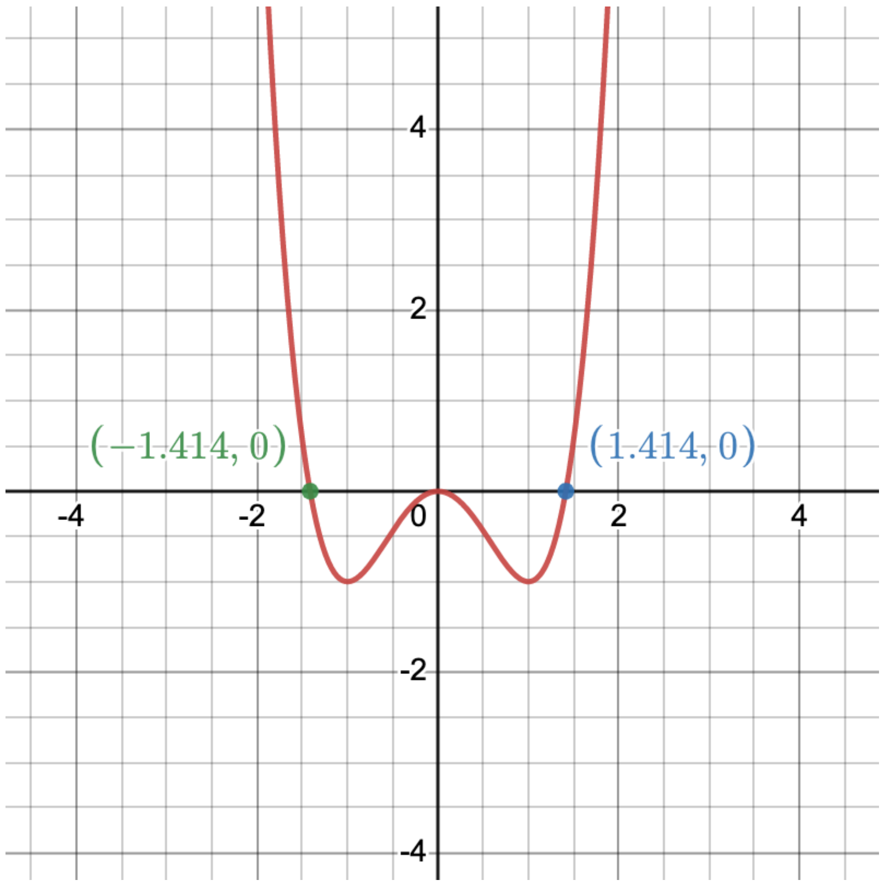
We determine $x = \pm\sqrt{2}$.

We find two real *Zeros*, namely $x = -\sqrt{2}$ and $x = \sqrt{2}$.

In summary, the *Zeros* are are $-\sqrt{2}$ (with multiplicity 1), are $\sqrt{2}$ (with multiplicity 1), and 0 (with multiplicity 2). The four *Zeros* are real numbers.

Example 12: Use *Zeros* of Polynomial Functions

Following is the graph of the polynomial function $f(x) = x^4 - 2x^2$ from Example 10. Note the end behavior given that the degree is even and the leading coefficient is positive!



As you can see, the real *Zeros* $-\sqrt{2} \approx -1.414$, 0, and $\sqrt{2} \approx 1.414$ are associated with the *x*-intercepts of the graph of the function.

Since 0 is of EVEN multiplicity, the graph touches the *x*-axis mimicking a parabola.

Since $-\sqrt{2}$ and $\sqrt{2}$ are of multiplicity 1, the graph crosses the *x*-axis in a straight line.

Example 13: Find *Zeros* of Polynomial Functions (1 of 3)

Find the real *Zeros* of $g(x) = P(x) = (x - 5)^2(x + 2)^4$. Give their multiplicity and state if the graph crosses the *x*-axis or touches the *x*-axis at each *Zero*.

This function is written as a product of factors. Specifically, the factors are of the form $\boldsymbol{(x - r)^m}$. When this is the case, we can find the degree of the function by adding the exponents. That is, 2 + 4 = 6. It tells us that we must find 6 *Zeros*.

Step 1 - We find the *Zeros* of *P* by setting $g(x)$ *equal* to 0 and solving the resulting equation.

$$(x - 5)^2(x + 2)^4 = 0$$

Step 2 - The function is already written as a product of factors.

Example 13: Find *Zeros* of Polynomial Functions (2 of 3)

Step 3 - We can now attempt to use the *Zero Product Principle* to solve the equation for x.

We get $(x - 5)^2 = 0$ and $(x + 2)^4 = 0$, and both equations can easily be solved.

Given $(x - 5)^2 = 0$, we use the Square Root Property to solve for x.

We get $x - 5 = \pm\sqrt{0} = 0$. (There is no negative 0!)

Then $x = 0 + 5$ and $x = 5$.

However, we must actually state that 5 is a *Zero* of multiplicity 2.

Example 13: Find *Zeros* of Polynomial Functions

Given $(x + 2)^4 = 0$, we will simply take the fourth root of both sides of the equal sign to solve for x.

Please note that taking roots of EVEN index of both sides of the equal sign requires a $\pm$ in the solution.

We get $x + 2 = \pm\sqrt[4]{0} = 0$. (There is no negative 0!)

Then $x = 0 - 2$ and $x = -2$.

However, we must actually state that – 2 is a *Zero* of multiplicity 4.

In summary, the *Zeros* are 5 (with multiplicity 2) and – 2 (with multiplicity 4). The six *Zeros* are real numbers.

Example 14: Use *Zeros* of Polynomial Functions

The graph of the polynomial function $g(x) = P(x) = (x - 5)^2(x + 2)^4$ is impossible to show in a reasonable coordinate system!

Since 5 is of multiplicity 2, which is even, the graph of the function touches the *x*-axis at 5 mimicking a parabola  .

Furthermore, since – 2 is of multiplicity 4, which is also even, the graph of the function touches the *x*-axis at – 2 also mimicking a parabola  .