# Examples
# Rational Functions

Based on power point presentations by Pearson Education, Inc.
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# Learning Objectives

1. Define rational functions and find their domains.
2. Define vertical asymptotes.
3. Find vertical asymptotes.
4. Define horizontal asymptotes.
5. Find horizontal asymptotes.

# Example 1: Find the Domain of a Rational Function

Find the domain in *Set-Builder Notation* of the rational function $f(x) = \dfrac{x^2 - 25}{x - 5}$.

Because division by 0 is undefined, we must exclude from the domain of the function the values of *x* that cause the denominator to be 0.

We find these values to be excluded by setting the denominator equal to 0.

$x - 5 = 0$

$x = 5$

The domain of *f* consists of *All Real Numbers except 5*. In *Set-Builder Notation* that's $\{x | x \neq 5\}$ .

# Example 2: Find the Domain of a Rational Function

Find the domain in *Set-Builder Notation* of the rational function $g(x)=\dfrac{x}{x^2-25}$.

Because division by 0 is undefined, we must exclude from the domain of the function the values of *x* that cause the denominator to be 0.

We find these values to be excluded by setting the denominator equal to 0.

$$x^2-25=0$$

$$x^2=25$$

We used the *Square Root Property*!

$$x=\pm 5$$

The domain of *g* consists of *All Real Numbers except −5* and *5*. In *Set-Builder Notation* that's $\{x|x\neq -5,\ x\neq 5\}$.

# Example 3: Find the Domain of a Rational Function

Find the domain of the rational function $h(x) = \dfrac{x+5}{x^2+25}$.

Because division by 0 is undefined, we must exclude from the domain of the function the values of *x* that cause the denominator to be 0.

We find these values to be excluded by setting the denominator equal to 0.

$$x^2 + 25 = 0$$

$$x^2 = -25$$

$$x = \pm 5i$$

We used the *Square Root Property*!

No real numbers cause the denominator of *h*(*x*) to equal 0. Therefore, the domain of *h* consists of *All Real Numbers*. This is can be nicely written in *Interval Notation* as (−∞, ∞).

# Example 4: Find the Domain and Equations of Vertical Asymptotes (1 of 2)

Find the domain and the equation(s) of the vertical asymptote(s), if any, of the graph of the rational function

$$f(x)=\frac{x+3}{(x+2)(x-1)}$$

**Domain:**

We set the denominator equal to 0 and solve.

$(x + 2)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 2 = 0$.

We find $x = 1$ and $x = -2$, which we must exclude from the domain.

Therefore, the domain is $\{ x \mid x \neq -2 \text{ and } 1 \}$.

# Example 4: Find the Domain and Equations of Vertical Asymptotes (2 of 2)

**Vertical Asymptotes:**

Since there are no common factors in the numerator and the denominator, we find the equation(s) of the vertical asymptote(s) by setting the entire denominator equal to 0.

$(x + 2)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 2 = 0$.

We find $x = 1$ and $x = -2$.

Thus, the lines $x = 1$ and $x = -2$ are the equations of the vertical asymptotes of the graph of $f$.

Please note that the asymptotes happen at the two numbers that are excluded from the domain!

# Example 5: Find the Domain and Equations of Vertical Asymptotes (1 of 2)

Find the domain and the equation of the vertical asymptote(s), if any, of the graph of the rational function

$$g(x) = \frac{(x+3)(x-1)}{(x+2)(x-1)}$$

**Domain:**

We set the denominator equal to 0 and solve.

$(x + 2)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 2 = 0$.

We find $x = 1$ and $x = -2$, which we must exclude from the domain.

Therefore, the domain is $\{ x \mid x \neq -2 \text{ and } 1 \}$.

# Example 5: Find the Domain and Equations of Vertical Asymptotes (2 of 2)

**Vertical Asymptotes:**

Since there is a common factor in the numerator and the denominator, $(x - 1)$, we find the equation of the vertical asymptote by setting equal to 0 only the factor in the denominator that doesn't also show up in the numerator.

$x + 2 = 0$

We find $x = -2$.

Thus, the line $x = -2$ is the equation of the only vertical asymptote of the graph of $g$.

NOTE: If we set the factor $(x - 1)$ equal to 0, we find a hole in the graph, namely at $x = 1$.

# Example 6: Find Equations of Vertical Asymptotes (1 of 2)

Find the *vertical asymptotes*, if any, of the graph of the rational function

$$f(x) = \frac{x}{x^2 - 1}$$

First, let's fully factor the numerator and denominator relative to the integers, to determine if the function is reduced to lowest terms.

$$f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x+1)(x-1)}$$

The denominator is a difference of squares!

We see that the numerator and denominator have no factors in common. therefore, the function is reduced to lowest terms.

# Example 6: Find Equations of Vertical Asymptotes (2 of 2)

Since there are NO common factors in the numerator and the denominator, we find the equation(s) of the vertical asymptote(s) by setting the entire denominator equal to 0.

$(x + 1)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 1 = 0$.

We find $x = 1$ and $x = -1$.

Thus, the lines $x = 1$ and $x = -1$ are the equations of the vertical asymptotes in the graph of $f$.

# Example 7: Find Equations of Vertical Asymptotes (1 of 2)

Find the vertical asymptotes, if any, of the graph of the rational function

$$g(x) = \frac{x+1}{x^2-1}$$

First, let's fully factor the numerator and denominator relative to the integers to determine if the function is reduced to lowest terms.

$$g(x) = \frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)}$$

We notice that the numerator and denominator have one common factor, namely ($x$ + 1).

# Example 7: Find Equations of Vertical Asymptotes (2 of 2)

In order to find the equation(s) of the vertical asymptote(s), we must only set the factors equal to 0 that do not also appear in the numerator. Therefore, we only set $(x - 1)$ equal to 0 and calculate as follows:

$x - 1 = 0$

and $x = 1$, which is the equation of the only vertical asymptote!

NOTE: If we set the factor $(x + 2)$ equal to 0, we find a hole in the graph, namely at $x = -2$.

# Example 8: Find the Equation of a Horizontal Asymptote

Find the horizontal asymptote, if any, of the graph of the rational function

$$f(x) = \frac{15x}{3x^2+1}$$

The degree of the numerator is 1 and the degree of the denominator is 2. The equation of the horizontal asymptote is *y = 0*.

# Example 9: Find the Equation of a Horizontal Asymptote

Find the horizontal asymptote, if any, of the graph of the rational function

$$k(x) = \frac{9x^2}{3x^2 + 1}.$$

The degree of the numerator is 2. It is equal to the degree of the denominator.

The leading coefficients of the numerator and the denominator, 9 and 3, are used to obtain the equation of the horizontal asympto*te*.

The equation of the horizontal asymptote is $y = \frac{9}{3}$ or $y = 3.$

# Example 10: Find the Equation of a Horizontal Asymptote

Find the horizontal asymptote, if any, of the graph of the rational function

$$h(x) = \frac{9x^3}{3x^2 + 1}.$$

The degree of the numerator is 3. It is greater than the degree of the denominator which is 2.

Thus, the graph of *h* has no horizontal asymptote.