Examples
Rational Functions

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Define rational functions and find their domains.
2. Memorize the characteristics of the graphs of rational functions.
3. Find the equations of vertical asymptotes.
4. Find the equations of horizontal asymptotes.
5. Memorize the domain and range, as well as the characteristics of the graph of the *Reciprocal Function*.

Example 1: Find the Domain of a Rational Function

Find the domain in *Set-Builder Notation* of the rational function $g(x) = \frac{x}{x^2 - 25}$.

Because division by 0 is undefined, we must exclude from the domain of any rational function the number replacements for *x* that cause the denominator to be 0.

We find these numbers by setting the denominator equal to 0 and solve for the variable.

$$x^2 - 25 = 0$$

$$x^2 = 25$$

We used the *Square Root Property*!

$$x = \pm 5$$

The domain of *g* consists of *All Real Numbers except −5* and *5*. In *Set-Builder Notation* that's $\{x | x \neq -5,\ x \neq 5\}$.

Example 2: Find the Domain of a Rational Function

Find the domain of the rational function $h(x) = \dfrac{x+5}{x^2+25}$.

We set the denominator equal to 0 and solve for the variable.

$$x^2 + 25 = 0$$

$$x^2 = -25$$ We used the *Square Root Property*!

$$x = \pm 5i$$

No real numbers cause the denominator to equal 0. Therefore, the domain consists of *All Real Numbers*. This is can be nicely written in *Interval Notation* as $(-\infty, \infty)$.

Example 3: Find the Domain and Equations of Vertical Asymptotes (1 of 2)

Find the domain and the equation(s) of the vertical asymptote(s), if any, of the graph of the rational function

$$f(x) = \frac{x+3}{(x+2)(x-1)}$$

Domain:

We set the denominator equal to 0 and solve.

$(x + 2)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 2 = 0$.

We find $x = 1$ and $x = -2$, which we must exclude from the domain.

Therefore, the domain is $\{ x \mid x \neq -2 \text{ and } 1 \}$.

Example 3: Find the Domain and Equations of Vertical Asymptotes (2 of 2)

Vertical Asymptote(s):

Since there are no common factors in the numerator and the denominator, we find the equation(s) of the vertical asymptote(s) by setting the entire denominator equal to 0.

$(x + 2)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 2 = 0$.

We find $x = 1$ and $x = -2$.

Thus, the lines $x = 1$ and $x = -2$ are the equations of the vertical asymptotes of the graph of f.

Please note that the asymptotes happen at the two numbers that are excluded from the domain!

Example 4: Find the Domain and Equations of Vertical Asymptotes (1 of 2)

Find the domain and the equation of the vertical asymptote(s), if any, of the graph of the rational function

$$g(x) = \frac{(x+3)(x-1)}{(x+2)(x-1)}$$

Domain:

We set the denominator equal to 0 and solve.

$(x + 2)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 2 = 0$.

We find $x = 1$ and $x = -2$, which we must exclude from the domain.

Therefore, the domain is $\{ x \mid x \neq -2 \text{ and } 1 \}$.

Example 4: Find the Domain and Equations of Vertical Asymptotes (2 of 2)

Vertical Asymptote(s):

Since there is a common factor in the numerator and the denominator, $(x - 1)$, we find the equation of the vertical asymptote by setting equal to 0 only the factor in the denominator that doesn't also show up in the numerator.

$x + 2 = 0$

We find $x = -2$.

Thus, the line $x = -2$ is the equation of the only vertical asymptote of the graph of g.

NOTE: If we set the factor $(x - 1)$ equal to 0, we find a hole in the graph, namely at $x = 1$.

Please note that the asymptote and the hole happen at the two numbers that are excluded from the domain!

Example 5: Find the Equations of the Asymptotes (1 of 3)

If possible, find equations of the asymptotes of the graphs of the rational function

$$f(x) = \frac{x}{x^2 - 1}$$

Vertical Asymptote(s):

Here we need to first investigate if the numerator and denominator have factors in common. We do this by trying to write both the numerator and the denominator as products of prime factors.

$$f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x+1)(x-1)}$$

The numerator is prime. The denominator is a difference of squares!

Now we see that the numerator and denominator have no factors in common. therefore, the function is reduced to lowest terms.

Example 5: Find the Equations of the Asymptotes (2 of 3)

Since there are NO common factors in the numerator and the denominator, we find the equation(s) of the vertical asymptote(s) by setting the entire denominator equal to 0.

$(x + 1)(x - 1) = 0$

By the *Zero Product Principle*, we can state $x - 1 = 0$ or $x + 1 = 0$.

We find $x = 1$ and $x = -1$.

Thus, the lines $x = 1$ and $x = -1$ are the equations of the vertical asymptotes in the graph of f.

Example 5: Find the Equation of the Asymptotes

Horizontal Asymptote:

The degree of the numerator of $f(x)=\dfrac{x}{x^2-1}$ is 1 and the degree of the denominator is 2.

Therefore, the equation of the horizontal asymptote is $y = 0$.

Example 6: Find the Equations of Vertical Asymptotes

If possible, find equations of the asymptotes of the graphs of the rational function

$$g(x) = \frac{x+1}{x^2-1}$$

Vertical Asymptote(s):

Here we need to first investigate if the numerator and denominator have factors in common. We do this by trying to write both the numerator and the denominator as products of prime factors.

$$g(x) = \frac{x+1}{x^2-1} = \frac{(x+1)}{(x+1)(x-1)}$$

The numerator is prime. The denominator is a difference of squares!

Now we see that the numerator and denominator have one common factor, namely (x + 1).

Example 6: Find the Equations of Vertical Asymptotes (2 of 3)

To find the equation(s) of the vertical asymptote(s), we must only set the factors equal to 0 that do not also appear in the numerator. Therefore, we only set $(x - 1)$ equal to 0 and calculate as follows:

$x - 1 = 0$

and $x = 1$, which is the equation of the only vertical asymptote!

NOTE: If we set the factor $(x + 2)$ equal to 0, we find a hole in the graph, namely at $x = -2$.

Example 6: Find the Equation of the Asymptotes (3 of 3)

Horizontal Asymptote:

The degree of the numerator of $g(x) = \frac{x+1}{x^2-1}$ is 1 and the degree of the denominator is 2.

Therefore, the equation of the horizontal asymptote is $y = 0$.

Example 7: Find the Equations of Vertical Asymptotes (1 of 3)

If possible, find equations of the asymptotes of the graphs of the rational function

$$f(x) = \frac{15x}{3x^2 + 1}$$

Vertical Asymptote(s):

Here we need to first investigate if the numerator and denominator have factors in common. We do this by trying to write both the numerator and the denominator as products of prime factors.

All we can do with the numerator is write $3 \cdot 5 \cdot x$. The quadratic expression $3x^2 + 2$ contains two terms. The terms have no factors in common and the expression is not a *Difference of Squares*. Therefore, we can state that the denominator is prime.

We can conclude that the numerator and denominator have no factor in common.

Example 7: Find the Equations of Vertical Asymptotes (2 of 3)

To find the equation(s) of the vertical asymptote(s), we must only set the denominator equal to 0.

$3x^2 + 2 = 0$

We will use the *Square Root Property* to solve this quadratic equation.

$3x^2 = -2$

$x^2 = -\frac{2}{3}$

$x = \pm\sqrt{-\frac{2}{3}}$

Since we ended up with an imaginary number, we can state that the graph of the given rational function has NO *vertical asymptotes*.

Example 7: Find the Equations of Vertical Asymptotes (2 of 3)

Horizontal Asymptote:

The degree of the numerator of $f(x) = \dfrac{15x}{3x^2 + 1}$ is 1 and the degree of the denominator is 2. The equation of the horizontal asymptote is $y = 0$.

Example 8: Find the Equations of Vertical Asymptotes

If possible, find equations of the asymptotes of the graphs of the rational function

$$k(x) = \frac{9x^2}{3x^2 + 1}$$

Vertical Asymptote(s):

Here we need to first investigate if the numerator and denominator have factors in common. We do this by trying to write both the numerator and the denominator as products of prime factors.

All we can do with the numerator is write $3 \cdot 3 \cdot x \cdot x$. The quadratic expression $3x^2 + 2$ contains two terms. The terms have no factors in common and the expression is not a *Difference of Squares*. Therefore, we can state that the denominator is prime.

We can conclude that the numerator and denominator have no factor in common.

Example 8: Find the Equations of Vertical Asymptotes (2 of 3)

To find the equation(s) of the vertical asymptote(s), we must only set the denominator equal to 0.

$3x^2 + 2 = 0$

We will use the *Square Root Property* to solve this quadratic equation.

$3x^2 = -2$

$x^2 = -\frac{2}{3}$

$x = \pm\sqrt{-\frac{2}{3}}$

Since we ended up with an imaginary number, we can state that the graph of the given rational function has NO *vertical asymptotes*.

Example 8: Find the Equations of Vertical Asymptotes (3 of 3)

Horizontal Asymptote:

The degree of the numerator of $k(x)=\frac{9x^2}{3x^2+1}$ is 2. It is equal to the degree of the denominator.

The leading coefficients of the numerator and the denominator, 9 and 3, are used to obtain the equation of the horizontal asymptot*e*.

The equation of the horizontal asymptote is $y=\frac{9}{3}$ or $y=3$.

Example 9: Find the Equations of Vertical Asymptotes (1 of 3)

If possible, find equations of the asymptotes of the graphs of the rational function

$$h(x) = \frac{9x^3}{3x^2 + 1}$$

Vertical Asymptote(s):

Here we need to first investigate if the numerator and denominator have factors in common. We do this by trying to write both the numerator and the denominator as products of prime factors.

All we can do with the numerator is write $3 \cdot 3 \cdot x \cdot x \cdot x$. The quadratic expression $3x^2 + 2$ contains two terms. The terms have no factors in common and the expression is not a *Difference of Squares*. Therefore, we can state that the denominator is prime.

We can conclude that the numerator and denominator have no factor in common.

Example 9: Find the Equations of Vertical Asymptotes (2 of 3)

To find the equation(s) of the vertical asymptote(s), we must only set the denominator equal to 0.

$3x^2 + 2 = 0$

We will use the *Square Root Property* to solve this quadratic equation.

$3x^2 = -2$

$x^2 = -\frac{2}{3}$

$x = \pm\sqrt{-\frac{2}{3}}$

Since we ended up with an imaginary number, we can state that the graph of the given rational function has NO *vertical asymptotes*.

Example 9: Find the Equations of Vertical Asymptotes (3 of 3)

Horizontal Asymptote:

The degree of the numerator of $h(x)=\dfrac{9x^3}{3x^2+1}$ is 3. It is greater than the degree of the denominator which is 2.

Thus, the graph of *h* has NO horizontal asymptote.

Example 10: Graph the Reciprocal Function by Hand (1 of 4)

Graph the Reciprocal Function $f(x) = \frac{1}{x}$ by hand.

1. **Equation of its Vertical Asymptote:**

   Since the function is reduced to lowest terms, we set the denominator equal to 0 and solve.

   $\boldsymbol{x = 0}$ This is the equation of the *y*-axis.

2. **Equation of its Horizontal Asymptote:**

   We notice that the degree of the numerator is 0 and that of the denominator is 1. When the degree of the numerator is smaller than that of the denominator the equation of the horizontal asymptote is $\boldsymbol{y = 0}$.

   This is the equation of the *x*-axis.

Example 10: Graph the Reciprocal Function by Hand (2 of 4)

3. Point associated with the *y*-intercept (when *x* = 0):

$f(0) = \frac{1}{0}$ The *y*-value is undefined at *x* = 0.

This means that there is no *y*–intercept.

4. Point associated with the *x*-intercept (when *y* = 0):

$0 = \frac{1}{x}$

When multiplying by sides by *x* we get 0 = 1. This is a false statement.

This means that there is no *x*-intercept.

Example 10: Graph the Reciprocal Function by Hand (3 of 4)

5. Find additional points to either side of the vertical asymptote (*y*-axis).

How about $x = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1$, and 2?

Using $f(x) = \frac{1}{x}$, we get the following *y*-values:

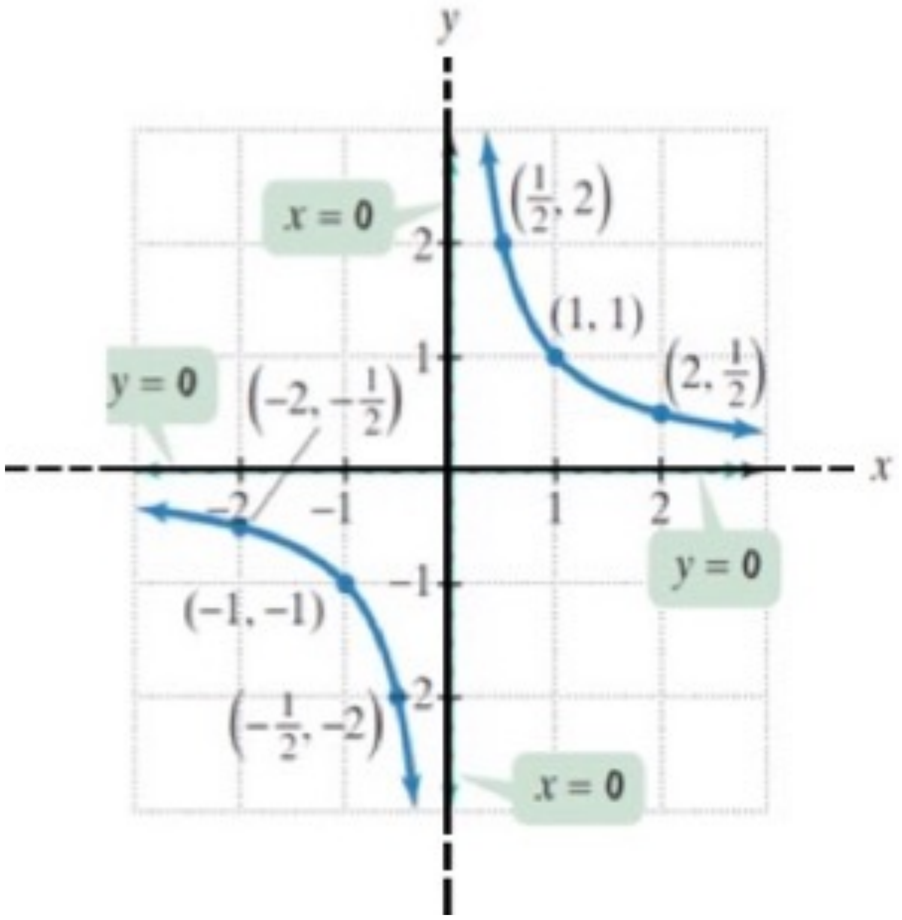
$f(-2) = \frac{1}{-2} = -\frac{1}{2}$ $\quad f(-1) = \frac{1}{-1} = -1$ $\quad f\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}} = -2$*

$f\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{2}} = 2$* $\quad f(1) = \frac{1}{1} = 1$ $\quad f(2) = \frac{1}{2}$

* Please note that dividing by a fraction is the same as multiplying by its reciprocal!

Example 10: Graph the Reciprocal Function by Hand (4 of 4)

6. Connect all points found in the previous steps keeping in mind the shape of the graph.



Since the horizontal asymptote is the *x*-axis and the vertical asymptote the *y*-axis, we will not graph them as a dashed line.