Examples
Solve Systems of Linear Equations Using Matrices

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Define augmented matrices and write one for a system of linear equations.
2. Transform augmented matrices to row-echelon form.
3. Solve systems of linear equations using the matrix method.

Example 1: Perform a Matrix Row Operation

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and interchange Row 1 and Row 2.

We will write these instructions as $R_1 \leftrightarrow R_2$ in between Row 1 and Row 2.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \end{matrix} = \left[\begin{array}{ccc|c} 1 & 6 & -3 & 7 \\ 4 & 12 & -20 & 8 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

Example 2: Perform a Matrix Row Operation

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and change the first element in Row 1 from 4 to 1.

Multiply all elements in Row 1 by $\frac{1}{4}$. We will write the instructions as $\frac{1}{4}R_1$ next to Row 1.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} \frac{1}{4}(4) & \frac{1}{4}(12) & \frac{1}{4}(-20) & \frac{1}{4}(8) \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

Example 3: Perform a Matrix Row Operation (1 of 2)

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and change the first element in Row 3 from – 3 to 0.

Replace Row 3 with the following sum: $3R_2 + R_3$

This means to multiply every element in Row 2 by 3 then add this product to the corresponding element in Row 3. We will write these instructions next to Row 3.

Example 3: Perform a Matrix Row Operation (2 of 2)

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] 3R_2 + R_3 = \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ \underset{3R2}{3(1)} + \underset{R3}{(-3)} & \underset{3R2}{3(6)} + \underset{R3}{(-2)} & \underset{3R2}{3(-3)} + \underset{R3}{1} & \underset{3R2}{3(7)} + \underset{R3}{(-9)} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 0 & 16 & -8 & 12 \end{array}\right]$$

Note that Row 3 was changed from [– 3   – 2   1 | – 9] to [0   16   – 8 | 12].

Example 4: Use Gauss-Jordan Elimination (1 of 6)

Solve the following system of two linear equations in two variables using Gauss-Jordan elimination:

$$\begin{cases} 2x + y = 1 \\ 3x + 2y = 4 \end{cases}$$

First, we will change to augmented matrix form as follows:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right]$$

Note that we only used the coefficients of the variables with the coefficients for *x* appearing in column 1, the coefficients for *y* in column 2, and the constants in column 3!

Please note that we also could have used the *Substitution Method* or the *Addition Method* to solve this system.

Example 4: Use Gauss-Jordan Elimination (2 of 6)

The next step is to change the augmented matrix to row-echelon form using allowable row operations.

$$\left[\begin{array}{cc|c} 1_1 & 0_4 & A \\ 0_2 & 1_3 & B \end{array}\right]$$

The first thing we must do is produce the number 1 in the first position of Row 1.

NOTE: It is easy to achieve 1’s because all we have to do is divide all elements of a row by the appropriate number.

Example 4: Use Gauss-Jordan Elimination (3 of 6)

We will first copy Row 2. Then we replace each element in Row 1 with the following calculation: $\frac{1}{2}R_1$

NOTE: To not get confused, it is best to write the instructions next to the row you want to change.

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \end{array} = \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 3 & 2 & 4 \end{array}\right]$$

Example 4: Use Gauss-Jordan Elimination (4 of 6)

The next thing we want is the number 0 in the first position of Row 2.

We will first copy Row 1. Then we replace each element in Row 2 with the following calculation: $-3R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 3 & 2 & 4 \end{array}\right]_{-3R_1+R_2} = \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \end{array}\right]$$

We calculated the terms of the new Row 2 as follows:

first term: $-3(1) + 3 = 0$

second term: $-3\left(\frac{1}{2}\right)+2=\frac{1}{2}$

constant: $-3\left(\frac{1}{2}\right)+4=\frac{5}{2}$

Example 4: Use Gauss-Jordan Elimination (5 of 6)

Working with the changed matrix, we now want the number 1 in the second position of Row 2.

We will first copy Row 1. Then we replace each element in Row 2 with the following calculation: $2R_2$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \end{array}\right]_{2R_2} = \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 5 \end{array}\right]$$

Working with the changed matrix, we now want the number 0 in the second position of Row 1.

We will first copy Row 2. Then we replace each element in Row 1 with the following calculation: $-\frac{1}{2}R_2 + R_1$

Example 4: Use Gauss-Jordan Elimination (6 of 6)

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 5 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2 + R_1} = \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

We calculated the terms of the new Row 1 as follows:

first term: $-\frac{1}{2}(0)+1=1$ second term: $-\frac{1}{2}(1)+\frac{1}{2}=0$ constant: $-\frac{1}{2}(5)+\frac{1}{2}=-2$

The original augmented matrix is now in row-echelon form. From it we can pick the results for ***x*** and ***y*** , namely

$x = -2$
$y = 5$

The solution to the given system is (– 2 , 5). Graphically this means that the two lines, defined by the two equations in the system, intersect at the point (– 2 , 5).

Example 5: Use Gauss-Jordan Elimination (1 of 5)

Solve the following system of two linear equations in two variables using Gauss-Jordan elimination:

$$\begin{cases} -x+4y=5 \\ x-y=1 \end{cases}$$

First, we will change to augmented matrix form as follows:

$$\left[\begin{array}{cc|c} -1 & 4 & 5 \\ 1 & -1 & 1 \end{array}\right]$$

The next step is to change the augmented matrix to row-echelon form using allowable row operations.

$$\left[\begin{array}{cc|c} 1_1 & 0_4 & A \\ 0_2 & 1_3 & B \end{array}\right]$$

Example 5: Use Gauss-Jordan Elimination (2 of 5)

We will first copy Row 2. Then we will replace each element in Row 1 with the following calculation: $-1R_1$

$$\left[\begin{array}{cc|c} -1 & 4 & 5 \\ 1 & -1 & 1 \end{array}\right]^{-1R_1} = \left[\begin{array}{cc|c} 1 & -4 & -5 \\ 1 & -1 & 1 \end{array}\right]$$

The next thing we want is the number 0 in the first position of Row 2.

We will first copy Row 1. Then we replace each element in Row 2 with the following calculation: $-1R_1 + R_2$

Example 5: Use Gauss-Jordan Elimination (3 of 5)

$$\left[\begin{array}{cc|c} 1 & -4 & -5 \\ 1 & -1 & 1 \end{array}\right] -1R_1 + R_2 = \left[\begin{array}{cc|c} 1 & -4 & -5 \\ 0 & 3 & 6 \end{array}\right]$$

We calculated the terms of the new Row 2 as follows:

first term: $-1(1) + 1 = 0$ second term: $-1(-4) + (-1) = 3$

constant: $-1(-5) + 1 = 6$

Example 5: Use Gauss-Jordan Elimination (4 of 5)

Working with the changed matrix, we now want the number 1 in the second position of Row 2.

We will first copy Row 1. Then we replace each element in Row 2 with the following calculation: $\frac{1}{3}R_2$

$$\left[\begin{array}{cc|c} 1 & -4 & -5 \\ 0 & 3 & 6 \end{array}\right] \begin{array}{c} \\ \frac{1}{3}R_2 \end{array} = \left[\begin{array}{cc|c} 1 & -4 & -5 \\ 0 & 1 & 2 \end{array}\right]$$

Working with the changed matrix, we now want the number 0 in the second position of Row 1.

We will first copy Row 2. Then we replace each element in Row 1 with the following calculation: $4R_2 + R_1$

Example 5: Use Gauss-Jordan Elimination (5 of 5)

$$\left[\begin{array}{cc|c} 1 & -4 & -5 \\ 0 & 1 & 2 \end{array}\right] \; 4R_2 + R_1 \quad = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

We calculated the terms of the new Row 1 as follows:

first term: $4(0) + 1 = 1$ second term: $4(1) + (-4) = 0$ constant: $4(2) + (-5) = 3$

The original augmented matrix is now in row-echelon form. From it we can pick the results for ***x*** and ***y***, namely

$x = 3$
$y = 2$

The solution to the given system is (3, 2). Graphically this means that the two lines, defined by the two equations in the system, intersect at the point (3, 2).

Example 6: Use Gauss-Jordan Elimination (1 of 11)

Solve the following system of three linear equations in three variables using Gauss-Jordan elimination:

$$\begin{cases} -2x - 4y - 2z = -18 \\ -4x - y + 2z = 10 \\ 4x + 3y + 2z = 10 \end{cases}$$

First, we will change to augmented matrix form:

$$\left[\begin{array}{ccc|c} -2 & -4 & -2 & -18 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right]$$

Note that we only used the coefficients of the variables with the coefficients for *x* appearing in column 1, the coefficients for *y* in column 2, the coefficients for *z* in column 3, and the constants in column 4!

Example 6: Use Gauss-Jordan Elimination (2 of 11)

The next step is to change the augmented matrix to row-echelon form using allowable row operations.

$$\left[\begin{array}{ccc|c} 1_1 & 0_9 & 0_8 & A \\ 0_2 & 1_4 & 0_7 & B \\ 0_3 & 0_5 & 1_6 & C \end{array}\right]$$

The first thing we must do is produce the number 1 in the first position of Row 1. I t is easy to achieve 1's because all we have to do is divide all elements of a row by the appropriate number.

Example 6: Use Gauss-Jordan Elimination (3 of 11)

We will first copy Rows 2 and 3. Then we replace each element in Row 1 with the following calculation: $-\frac{1}{2}R_1$

NOTE: To not get confused, it is best to write the instructions next to the row you want to change!

$$\left[\begin{array}{ccc|c} -2 & -4 & -2 & -18 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right] \begin{array}{l} -\frac{1}{2}R_1 \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right]$$

Example 6: Use Gauss-Jordan Elimination (4 of 11)

The next thing we want is the number 0 in the first position of Row 2. This is not so easy to achieve. To get 0's we must always add two rows *R* (sometimes using multiplication) to replace the row in which we want the 0.

We will first copy Rows 1 and 3. Then, we replace each element in Row 2 with the following calculation: $4R_1 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right] \quad 4R_1 + R_2 \quad = \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 4 & 3 & 2 & 10 \end{array}\right]$$

Example 6: Use Gauss-Jordan Elimination (5 of 11)

Working with the changed matrix, we now want the number 0 in the first position of Row 3.

We will first copy Rows 1 and 2. Then we replace each element in Row 3 with the following calculation: $-4R_1 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 4 & 3 & 2 & 10 \end{array}\right]_{-4R_1 + R_3} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 0 & -5 & -2 & -26 \end{array}\right]$$

Example 6: Use Gauss-Jordan Elimination

Working with the changed matrix, we now want the number 1 in the second position of Row 2.

We will first copy Rows 1 and 3. Then we replace each element in Row 2 with the following calculation: $\frac{1}{7}R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 0 & -5 & -2 & -26 \end{array}\right] \frac{1}{7}R_2 \quad = \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & -5 & -2 & -26 \end{array}\right]$$

We ended up with some “nasty” fractions in Row 2!

Example 6: Use Gauss-Jordan Elimination (7 of 11)

Working with the changed matrix, we now want the number 0 in the second position of Row 3.

We will first copy Rows 1 and 2. Then we replace each element in Row 3 with the following calculation: $5R_2 + R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & -5 & -2 & -26 \end{array}\right] 5R_2 + R_3 \quad = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & \frac{16}{7} & \frac{48}{7} \end{array}\right]$$

We calculated the terms of the new Row 3 as follows:

first term: $5(0) + 0 = 0$

second term: $5(1) + (-5) = 0$

third term: $5\left(\frac{6}{7}\right) + (-2) = \frac{16}{7}$

constant: $5\left(\frac{46}{7}\right) + (-26) = \frac{48}{7}$

Example 6: Use Gauss-Jordan Elimination

Working with the changed matrix, we now want the number 1 in the third position of Row 3.

We will first copy Rows 1 and 2. Then we replace each element in Row 3 with the following calculation: $\frac{7}{16}R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & \frac{16}{7} & \frac{48}{7} \end{array}\right] \begin{array}{c} \\ \\ \frac{7}{16}R_3 \end{array} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example 6: Use Gauss-Jordan Elimination (9 of 11)

Working with the changed matrix, we now want the number 0 in the third position of Row 2.

We will first copy Rows 1 and 3. Then we replace each element in Row 2 with the following calculation: $-\frac{6}{7}R_3 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & 1 & 3 \end{array}\right] -\frac{6}{7}R_3 + R_2 \quad = \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example 6: Use Gauss-Jordan Elimination (10 of 11)

Working with the changed matrix, we now want the number 0 in the third position of Row 1.

We will first copy Rows 2 and 3. Then we replace each element in Row 1 with the following calculation: $-1R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]^{-1R_3 + R_1} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Example 6: Use Gauss-Jordan Elimination

Working with the changed matrix, we now want the number 0 in the second position of Row 1.

We will copy Rows 2 and 3. The we replace each element in Row 1 with the following calculation: $-2R_2+R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{c} -2R_2+R_1 \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The original augmented matrix is now in row-echelon form. From it we can pick the results for ***x***, ***y***, and ***z***, namely

$x = -2$ $\qquad$ $y = 4$ $\qquad$ $z = 3$

The solution to the given system is (– 2, 4, 3). Graphically this means that the three planes, defined by the three equations in the system, intersect at the point (– 2, 4, 3).

Example 7: Interpret a Solution Matrix

Discuss the following solution matrix of a system of three linear equations in three variables x, y, and z.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This matrix is in proper row-echelon form. The solutions for x, y, and z are

$x = -2$

$y = 4$

$z = 3$

Graphically, three planes intersect in exactly one point, namely (– 2, 4, 5).

Example 8: Interpret a Solution Matrix

Discuss the following solution matrix of a system of three linear equations in three variables *x*, *y*, and *z*.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

There is a 0 instead of a 1 in the third position of Row 3. This actually means 0 = – 2 which is, of course, a false statement. Therefore, we can say that this system has no solutions.

Graphically, any two or all three planes in 3D space could be parallel to each other.

Example 9: Interpret a Solution Matrix

Discuss the following solution matrix of a system of three linear equations in three variables *x*, *y*, and *z*.

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last row contains only 0s. There is not a 1 in the third position of Row 3. However, 0 = 0 is a true statement. This indicates that there are an infinite number of solutions to this system.

Graphically, the three planes in 3D space intersect in one line or they could all three be one and the same plane.