Concepts
Systems of Linear Equations in Three Variables

Based on power point presentations by Pearson Education, Inc.

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Learning Objectives

1. Draw a three-dimensional coordinate system.
2. Memorize the characteristics of a system of three linear equations in three variables.
3. Solve systems of three linear equations in three variables.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

1. The Three-Dimensional Coordinate System (1 of 3)

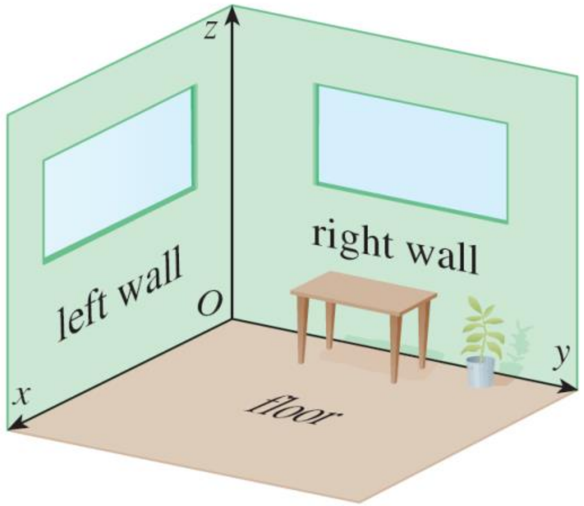
Before we discuss systems of three linear equations in three variables, let's investigate the three-dimensional coordinate system.

So far, we only discussed two-dimensional coordinate systems, which consist of a plane. We know that any point in the plane can be represented as an **ordered pair**, say (x, y), of real numbers.

In a three-dimensional coordinate system, three real numbers are required to pinpoint the location of a point. We represent this point by an **ordered triple,** say (x, y, z), of real numbers.

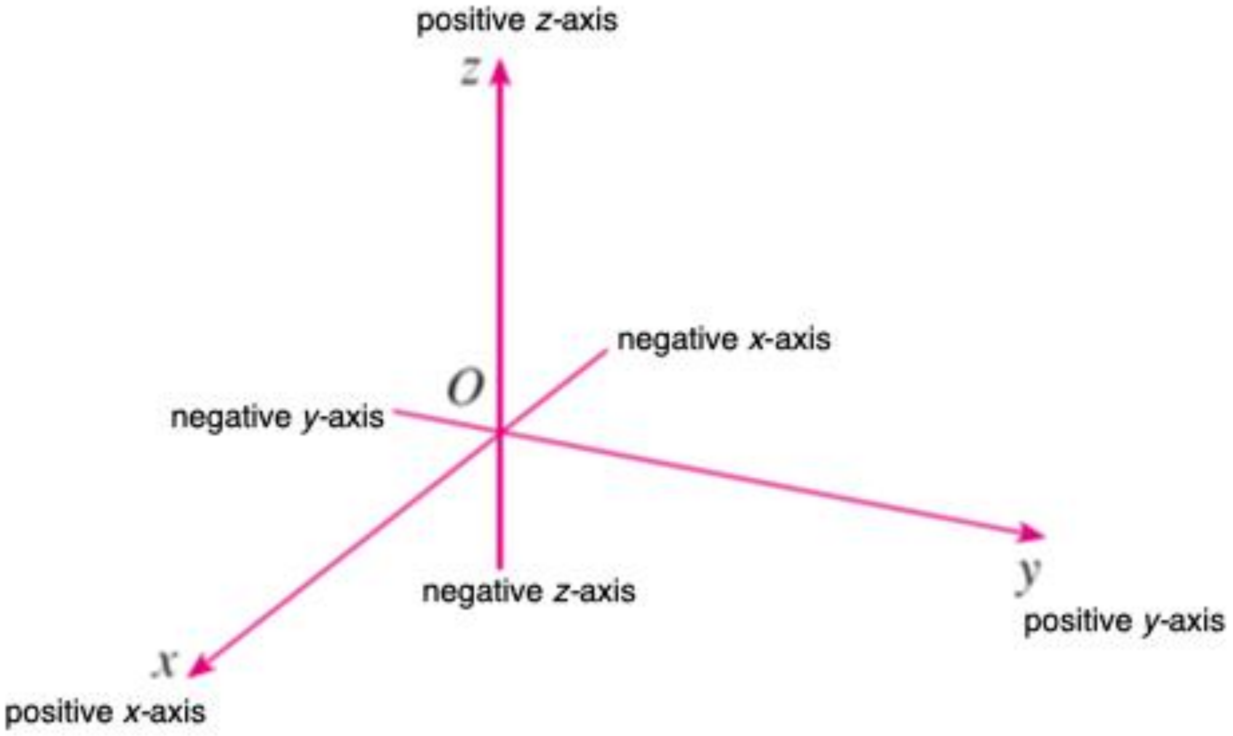
The Three-Dimensional Coordinate System (2 of 3)

Because many people have some difficulty visualizing three-dimensional coordinate systems, it might be helpful to look at any bottom corner of a room and call the corner the origin. The wall on the left is in the *xz*-plane, the wall on the right is in the *yz*-plane, and the floor is in the *xy*-plane.



The Three-Dimensional Coordinate System (3 of 3)

Following is an illustration of the positive and negative *x*-, *y*-, and *z*-axes.



2. Introduction to Systems of Three Linear Equations in Three Variables (1 of 2)

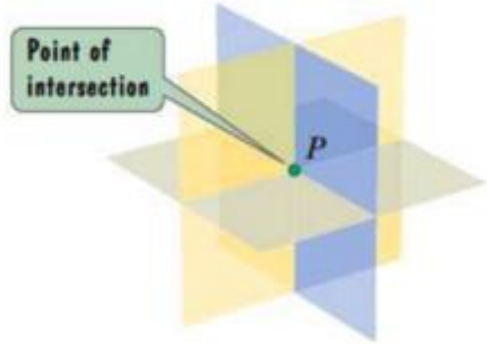
In general, any equation of the form $Ax + By + Cz + D = 0$ is considered a linear equation in three variables x, y, and z.

The graph of a linear equation in three variables is a PLANE in three-space. When solving a system of three equations in three variables, we are actually trying to find the point of intersection of the three planes.

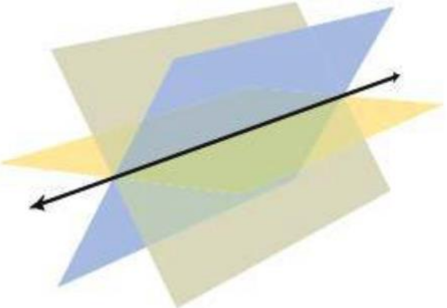
The solution of a system of three linear equations in three variables is an ordered triple (x, y, z) of real numbers.

Introduction to Systems of Three Linear Equations in Three Variables (2 of 2)

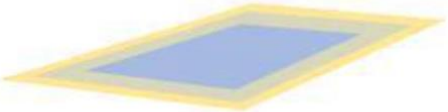
The following pictures show what types of solutions we can expect for a system of three linear equations in three variables.



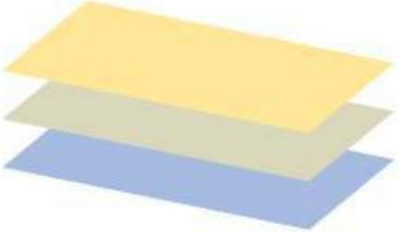
The planes intersect in a common point.



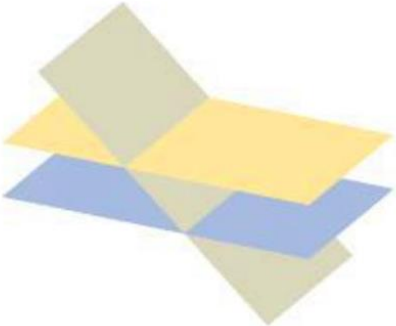
The planes intersect along a common line.



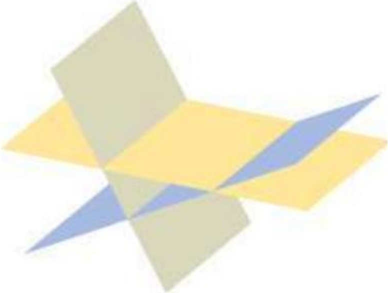
The planes coincide.



Three planes are parallel with no common intersection point.



Two planes are parallel with no common intersection point.



Planes intersect two at a time. There is no intersection point common to all three planes.

3. Solve Systems of Three Linear Equations in Three Variables

A system of three linear equations in three variables can be solved algebraically mainly by the *Elimination Method*. In this course, we are only interested in finding exactly one solution (one point of intersection).

Strategy to try to find a Point of Intersection of three Planes:

Step 1 - Unless already done, rewrite both equations in the system so that like terms are directly below each other with variable terms appearing on the left and constants appearing on the right side of the equal sign.

Example:

Solve the system $\begin{cases} x+4y-z=20 & \text{Equation 1} \\ 3x+2y+z=8 & \text{Equation 2} \\ 2x-3y+2z=-16 & \text{Equation 3} \end{cases}$

Step 1 is already done!

Solve Systems of Three Linear Equations in Three Variables

Step 2 -

(a) Select two original equations. Eliminate one of the variables by addition. We may have to first "appropriately" change the coefficients of one or both equations!

Example continued:

Let's select Equation 1 and Equation 2 in the system and eliminate the *z*-terms by addition.

$$\begin{array}{rl} x + 4y - z = 20 & \text{Equation 1} \\ + \quad \underline{3x + 2y + z = 8} & \text{Equation 2} \\ 4x + 6y \qquad = 28 & \text{(new) Equation 4} \end{array}$$

Solve Systems of Three Linear Equations in Three Variables

Step 2 -

(b) Use a different combination of two original equations and eliminate the SAME variable term that was eliminated in (a). We may have to first "appropriately" change the coefficients of one or both equations!

Example continued

Now we use Equations 1 and 3 in the system and eliminate the *z*-terms again by addition.

$x + 4y - z = 20$ Equation 1

$2x - 3y + 2z = -16$ Equation 3

Here we will first multiply all terms of Equation 1 by the number 2. Why?

Solve Systems of Three Linear Equations in Three Variables

Example continued:

The coefficients of *z* in the two equations are now opposite numbers and the *z*-terms are eliminated by addition!

$$
\begin{array}{rll}
 & 2x + 8y - 2z = 40 & 2 \cdot \text{Equation 1} \\
+ & \underline{2x - 3y + 2z = -16} & \text{Equation 3} \\
 & 4x + 5y \qquad = 24 & \text{(new) Equation 5}
\end{array}
$$

Solve Systems of Three Linear Equations in Three Variables

Step 3 – Create a system of two linear equations in two variables by using the two new equations found in Steps 2a and 2b. Then eliminate one of the variables by addition. We may have to first "appropriately" change the coefficients of one or both equations!

Example continued:

$$\begin{cases} 4x + 6y = 28 & \text{Equation 4} \\ 4x + 5y = 24 & \text{Equation 5} \end{cases}$$

We will eliminate the *x*-terms. However, we will need to multiply all terms in Equation 5 by the number – 1. Why?

Solve Systems of Three Linear Equations in Three Variables

Example continued:

The coefficients of *x* in the two equations are now opposite numbers and the *x*-terms are eliminated by addition!

$$\begin{array}{rrl} & 4x + 6y = 28 & \quad \text{Equation 4} \\ + & \underline{-4x - 5y = -24} & \quad -1 \cdot \text{Equation 5} \\ & y = 4 & \end{array}$$

This is the *y*-coordinate of the point of intersection!

Solve Systems of Three Linear Equations in Three Variables

Step 4 – If possible, substitute the value of the coordinate found in Step 3 into the "appropriate" variable in either one of the two NEW equations from Steps 2a and 2b. Solve the equation.

Example continued:

Let's use Equation 5 and substitute its y with $y = 4$ found in Step 3.

$4x + 5y = 24$ (Equation 5)

$4x + 5(4) = 24$
$4x + 20 = 24$
$4x = 4$
$x = 1$ This is the x-coordinate of the point of intersection!

Solve Systems of Three Linear Equations in Three Variables

Step 5 – If possible, substitute the values of the coordinates found in Steps 3 and 4 into the “appropriate” variables in either one of the ORIGINAL equations. Solve the equation.

Example continued:

Let’s use Equation 2 and replace its x and y with $y = 4$ (Step 3) and $x = 1$ (Step 4).

$3x + 2y + z = 8$ (Equation 2)

$3(1) + 2(4) + z = 8$

$3 + 8 + z = 8$

$11 + z = 8$

$z = -3$ This is the z-coordinate of the point of intersection!

Solve Systems of Three Linear Equations in Three Variables

Step 6 - Write the solution of the system as an ordered triple, if possible.

Example continued:

Given $x = 1$ (Step 4), $y = 4$ (Step 3), and $z = -3$ (Step 5), the solution of the system is as follows:

(1, 4, – 3)

Solve Systems of Three Linear Equations in Three Variables

Following is an illustration of the point created by the ordered triple (1, 4, – 3) in the three-dimensional coordinate system.