Concepts
Rational Functions

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Define rational functions and find their domains.
2. Memorize the characteristics of the graphs of rational functions.
3. Find the equations of vertical asymptotes.
4. Find the equations of horizontal asymptotes.
5. Memorize the domain and range, as well as the characteristics of the graph of the *Reciprocal Function*.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

1. Definition of Rational Functions (1 of 4)

We have already been exposed to rational equations where the variable was in the denominator. For example, $\frac{x-1}{4x}=0$.

Now, we will discuss rational equations in two variables where the independent variable x is in the denominator.

For example, $y=\frac{x-1}{4x}$.

Rational equations in two variables are functions, therefore, we can replace the dependent variable with function notation. For example, $h(x)=\frac{x-1}{4x}$.

Definition of Rational Functions (1 of 4)

The **general form** of the rational function in *x* is

$$f(x) = \frac{p(x)}{q(x)}$$, where ***p*** and ***q*** are polynomial functions and $\boldsymbol{q} \neq 0$.

Domain: *All Real Numbers EXCEPT those that make the denominator equal to 0* because division by 0 is undefined.

Definition of Rational Functions (2 of 4)

Examples of rational functions:

$$g(x) = \frac{x^2 + x - 6}{x^2 - 8x + 12}$$

$$k(x) = \frac{x - 6}{7}$$

The function ***k*** is actually a linear function involving fractions because it can be written as $k(x) = \frac{1}{7}x - \frac{6}{7}$.

NOTE: In this lesson, the rational functions we will be most interested in are the ones with a variable in the denominator.

Definition of Rational Functions (3 of 4)

Previously we were told that the domain of a rational function consists of *All Real Numbers EXCEPT those that make the denominator equal to 0* because division by 0 is undefined.

Since there are infinitely many different rational function, there is then no one standard domain.

Strategy for finding the domain of any rational number:

1. Set the denominator equal to 0 and solve for the variable.
2. Write the domain in Set-Builder Notation because Interval Notation can be cumbersome.

Definition of Rational Functions (4 of 4)

Example:

Find the domain in *Set-Builder Notation* of the rational function $f(x) = \dfrac{x^2 - 25}{x - 5}$.

We set the denominator equal to 0 and solve for the variable.

$x - 5 = 0$

$x = 5$

We find that we must exclude 5 from the domain. Specifically, the domain of *f* consists of *All Real Numbers except 5*.

The domain for the given function is $\{x | x \neq 5\}$ in *Set-Builder Notation*.

2. Graphs of Rational Functions (1 of 6)

In this course, we will not graph rational functions by hand. However, we must memorize certain features of their graphs.

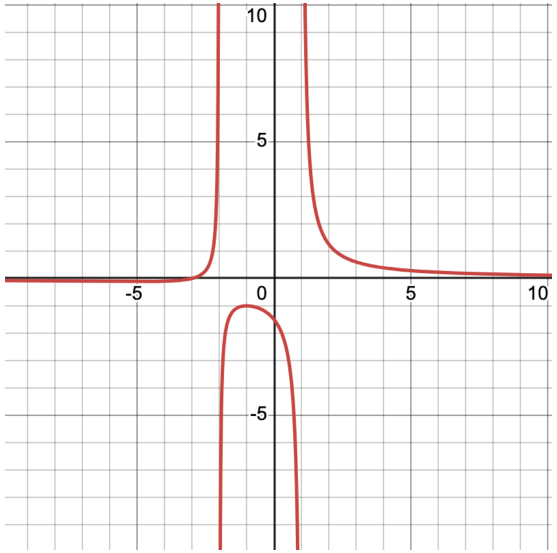
a. Some rational functions have graphs that are smooth and continuous. However, there are other rational functions whose graphs are NOT continuous.

b. Rational functions do not have one standard graph. Infinitely many different graphs are possible.

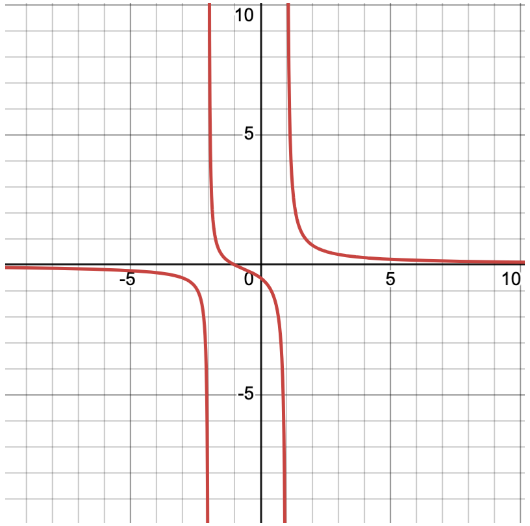
Graphs of Rational Functions (2 of 6)

Graphing utility drawn examples of graphs of rational functions.

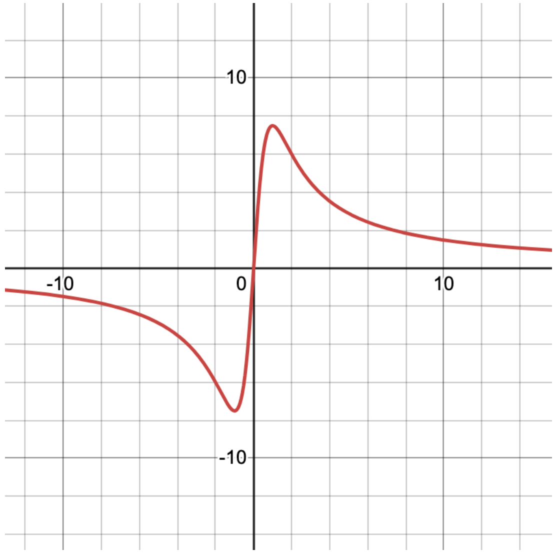
a.



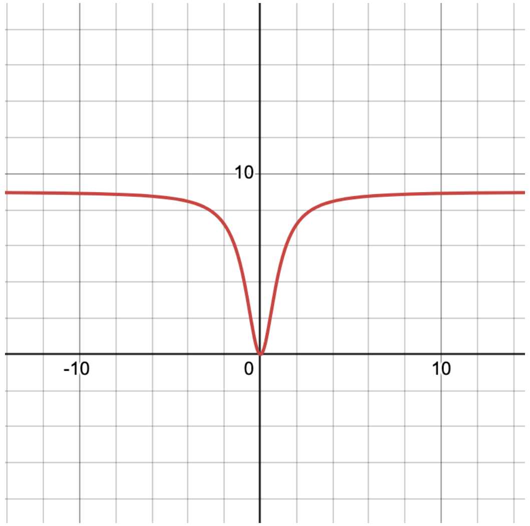
b.



c.



d.



Graphs of Rational Functions (3 of 6)

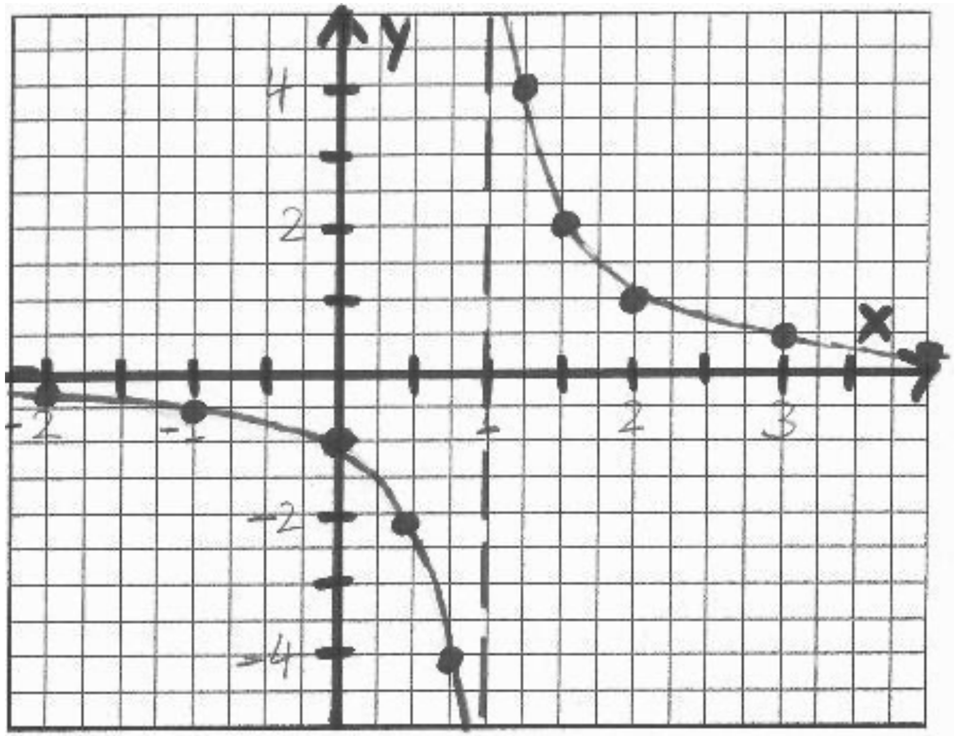
Since we often must exclude numbers from the domain of a rational function, the graph of the rational function is NOT continuous at these exclusions.

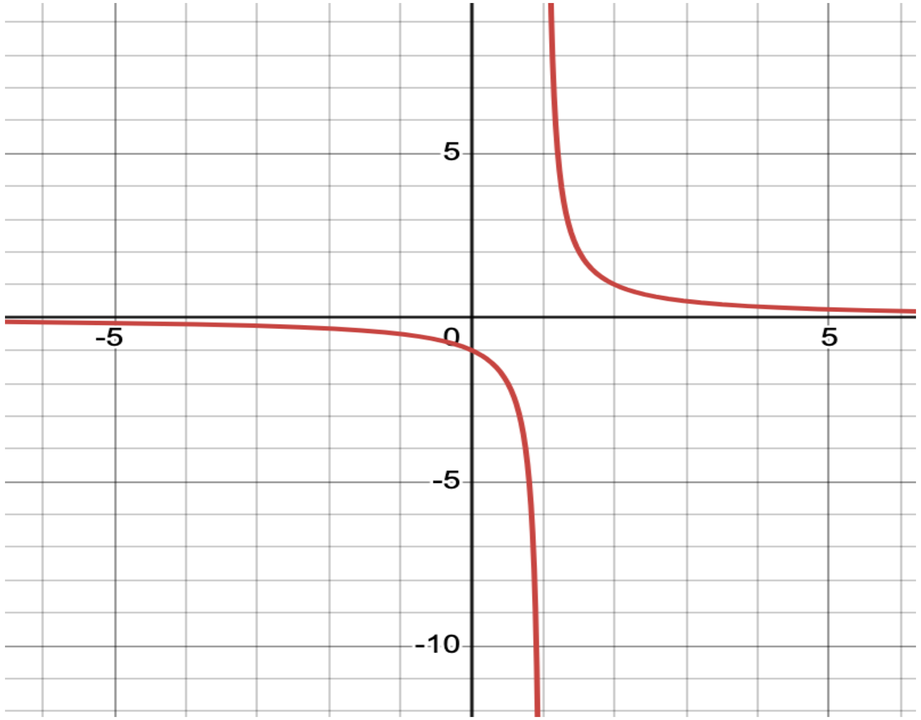
These “discontinuities” will show up either as **vertical asymptotes** or **holes** on the graph of a rational function.

Vertical asymptotes are invisible vertical lines. A graph can have infinitely many or there may not be any at all. *Vertical asymptotes* divide the graph of a rational function into two or more branches that will never meet. The graph of a rational function will never touch its *vertical asymptotes*, but it will continue to move toward them.

Graphs of Rational Functions (4 of 6)

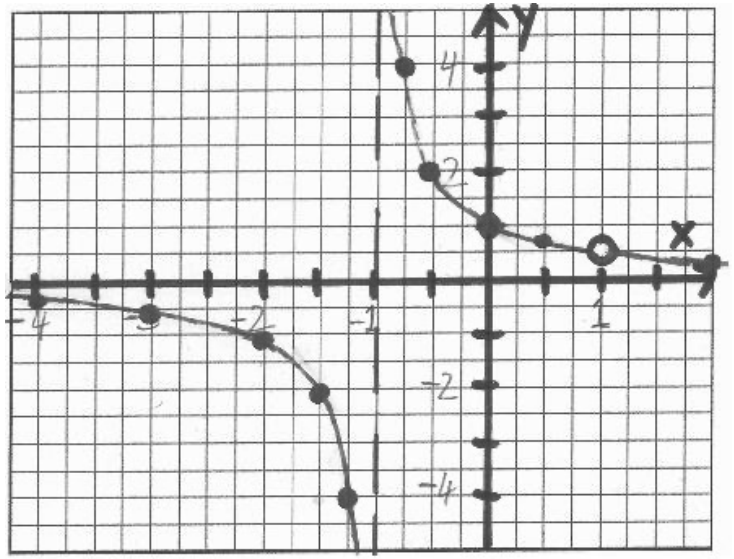
When we graph rational functions by hand, the *vertical asymptotes* are drawn as dashed vertical lines. This is supposed to indicate that they are not a visible part of the graph. However, when we graph rational functions with a graphing utility, the *vertical asymptotes* DO NOT show up on the graph.

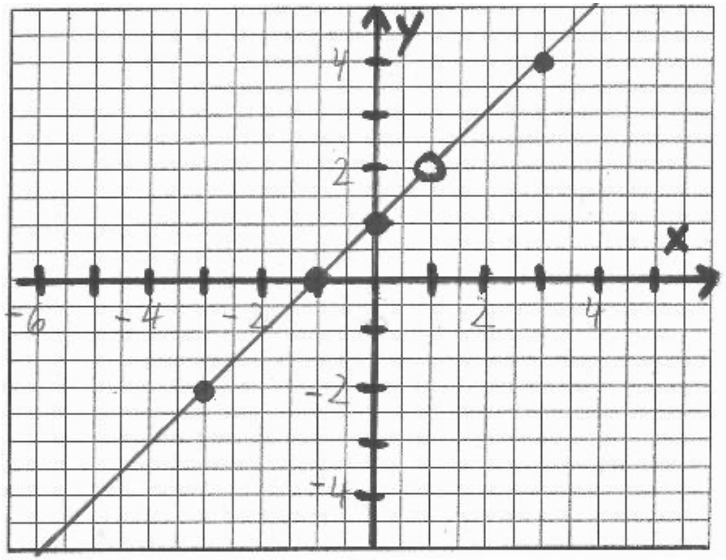




Graphs of Rational Functions (5 of 6)

The following graphs of rational functions have a **hole**. In a hand-drawn graph we indicate this with a circle. Unfortunately, when we graph such a rational function with a graphing utility, it cannot show the holes.





Please note that one graph has a *vertical asymptote* and a hole, while the other graph is a straight line with a hole. It does not have a *vertical asymptote*.

Graphs of Rational Functions (6 of 6)

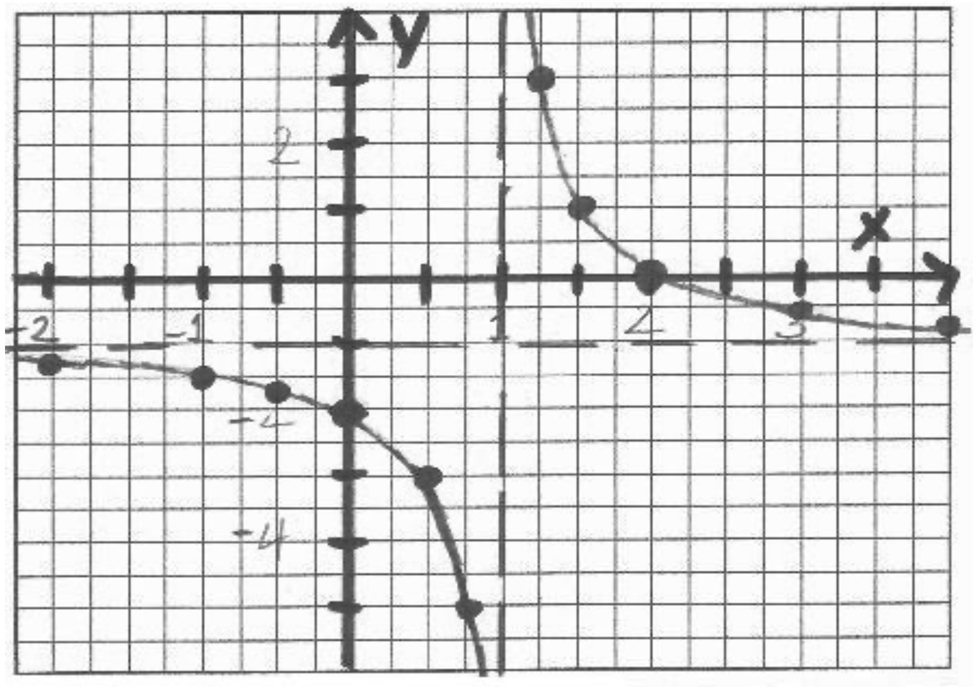
The graphs of some rational functions may also have a horizontal asymptote.

Horizontal asymptotes are invisible horizontal lines. A graph can have at most one or none. The graph might or might not touch its *horizontal asymptote*. If it does not touch it, it will continue to move toward it.

When we graph rational functions by hand, the *horizontal asymptotes* are drawn as dashed horizontal lines. This is supposed to indicate that they are not a visible part of the graph. However, when we graph rational functions with a graphing utility, the *horizontal asymptotes* DO NOT show up on the graph.

Graphs of Rational Functions (6 of 6)

Following is a hand-drawn graph of a rational function together with the same function being graphed with a graphing utility. We can clearly see a *horizontal* and *vertical asymptote* on the hand-drawn graph. They are not so obvious on the computer-generated one.



3. Find the Equations of Vertical Asymptotes (1 of 2)

1. If the numerator and denominator of a rational function have one or more factors in common, then we say that the rational function IS <u>NOT reduced to lowest terms</u>.

In this case, we find the vertical asymptotes by setting the factors in the denominator equal to 0 that do not also appear in the numerator. The solutions are then used to form equations of vertical lines which are the vertical asymptotes.

For example, the following function is NOT reduced to lowest terms:

$$g(x) = \frac{x(x-2)}{(x+1)(x-2)}$$ Domain: $\{ x \mid x \neq -1 \text{ and } 2\}$

The factor (x + 1) only shows up in the denominator. When we set it equal to 0 and solve, we get $x = -1$ which is the equation of the vertical asymptote.

The factor (x – 2) shows up BOTH in the numerator and in the denominator! When we set it equal to 0 and solve, we get $x = 2$ where 2 is the x-coordinate of the hole in the graph.

Find the Equations of Vertical Asymptotes (2 of 2)

2. If the numerator and denominator of a rational function have NO factors in common, we say that it is reduced to lowest terms.

   In this case, we set the entire denominator equal to 0 and solve. The solutions are then used to form equations of vertical lines which are the vertical asymptotes.

   For example, the following function is reduced to lowest terms:

   $$f(x) = \frac{x}{(x+1)(x-2)}$$ Domain: $\{ x \mid x \neq -1 \text{ and } 2\}$

   The numerator and the denominator have no factors in common! Therefore, we set both factors in the denominator equal to 0. We get $x = -1$ and at $x = 2$ which are the equations of the vertical asymptotes.

4. Find the Equations of Horizontal Asymptotes (1 of 2)

We find the equations of horizontal asymptotes by applying some calculus concepts which leads to the following three theorems.

Given a rational function $f(x)=\dfrac{p(x)}{q(x)}$ assume the following:

- the degree of the polynomial p is $\boldsymbol{n}$ and the degree of the polynomial q is $\boldsymbol{m}$.
- the leading coefficient of p is $\boldsymbol{a_n}$ and the leading coefficient of q is $\boldsymbol{b_m}$.

1. If $n < m$, the equation of the horizontal asymptote is $y = 0$ (x-axis).

   Example: Given $g(x)=\dfrac{x^2+1}{4x^3-4x+9}$, we can see that $n = 2$ and $m = 3$.

   Therefore, the equation of this horizontal asymptote is $y = 0$ (x-axis).

Find the Equations of Horizontal Asymptotes (2 of 2)

2. If $n = m$, the equation of the horizontal asymptote is $y = \frac{a_n}{b_m}$.

   Example: Given $h(x) = \frac{x-1}{4x}$ we can see that $n = 1$ and $m = 1$.

   The leading coefficient in the numerator is 1 and the one in the denominator is 4.

   Therefore, the equation of this horizontal asymptote is $y = \frac{1}{4}$.

3. If $n > m$, the graph has NO horizontal asymptote.

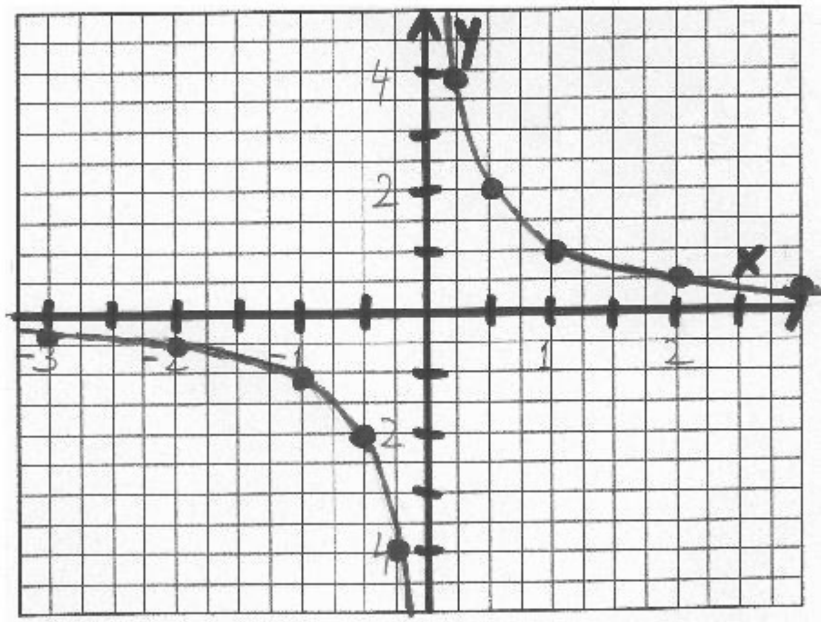
   Example: Given $p(x) = \frac{4x^3 - 4x + 9}{x^2 + 1}$, we can see that $n = 3$ and $m = 2$.

   Therefore, the graph has NO horizontal asymptote.

5. The Reciprocal Function

The *Reciprocal Function* is a special rational function which is sometimes called the *common rational function*. Following is its equation.

$$f(x) = \frac{1}{x}$$



Domain: $(-\infty,0)\cup(0,\infty)$ or $\{x \mid x \neq 0\}$

Basic Characteristics of the Graph:

- **The *y*-axis is the vertical asymptote**. The graph gets closer and closer to it but never touches it.
- **The *x*-axis is the horizontal asymptote**. The graph gets closer and closer to it but never touches it.
- There are no *x*- and *y*-intercepts.

NOTE: When the axes are asymptotes, we do not dash them!