Concepts
Rational Functions

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Define rational functions and find their domains.
2. Define vertical asymptotes.
3. Find vertical asymptotes.
4. Define horizontal asymptotes.
5. Find horizontal asymptotes.

1. Definition of Rational Functions (1 of 2)

Rational functions are quotients of polynomial functions. This means that rational functions can be expressed as

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions and $q(x) \neq 0$.

The domain of a rational function is the set of *All Real Numbers EXCEPT the x-values that make the denominator equal to 0.*

We find the values to be excluded from the domain by setting the denominator equal to 0 and solving the equation.

Definition of Rational Functions (2 of 2)

Examples of Rational Functions:

$$h(x) = \frac{x-1}{4x} \qquad k(x) = \frac{x^2+1}{4x^3-4x+9} \qquad p(x) = \frac{4x^3}{x^2+9} \qquad f(x) = x^2+2 \qquad g(x) = \frac{x-1}{7}$$

The function **f** is actually a quadratic function. However, it can be written as $f(x) = \frac{x^2+2}{1}$ which makes it a rational function by definition.

The function **g** is actually a linear function because it can be written as $g(x) = \frac{1}{7}x - \frac{1}{7}$ which is the slope-intercept form.

The rational functions we will be most interested in are the ones with a variable in the denominator like the first two.

2. Definition of Vertical Asymptotes (1 of 2)

The definition of rational functions indicates that we must exclude numbers from the domain which will make the denominator equal to 0.

This means, that at these excluded numbers, the graph of the rational function is discontinuous (not continuous).

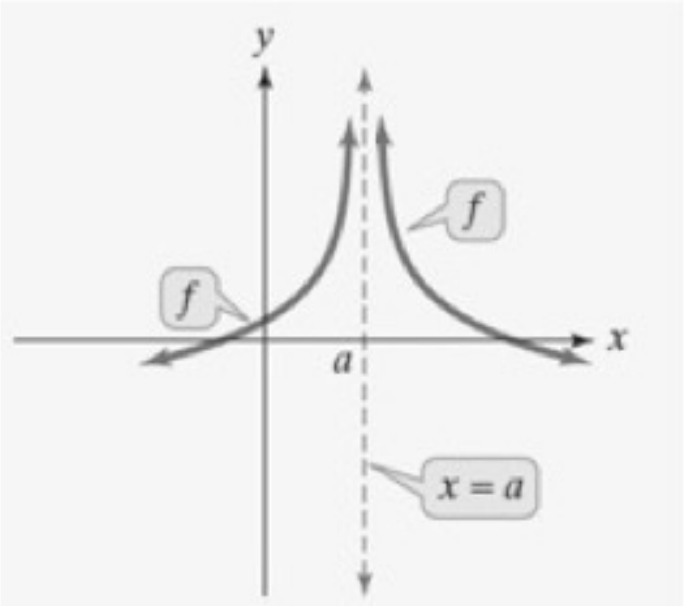
The discontinuities will show up either as **vertical asymptotes** or **holes** on the graph of a rational function.

Vertical asymptotes are invisible vertical lines. There can be infinitely many, and they divide the graph of the rational function into two or more branches.

Definition of Vertical Asymptotes (2 of 2)

The dashed line $x = a$ in the picture of a rational function below is a vertical asymptote of the graph of the function f. **The graph of a rational function will never touch its vertical asymptotes, but it will continue to move toward them**.

Depending on the makeup of the rational function, its graph may not have a vertical asymptote!



When we graph rational functions by hand, the vertical asymptotes are drawn as dashed lines. This is supposed to indicate that they are not a visible part of the graph.
When we graph rational functions with a graphing utility, the vertical asymptotes DO NOT show up on the graph.

3. Find the Equations of Vertical Asymptotes (1 of 2)

1. If the numerator and denominator of a rational function have one or more factors in common, then we say that the rational function IS <u>NOT reduced to lowest terms</u>.

   In this case, we find the vertical asymptotes by setting the factors in the denominator equal to 0 that do not also appear in the numerator. The solutions are then used to form equations of vertical lines which are the vertical asymptotes.

   For example, the following function is NOT reduced to lowest terms:

   $$g(x) = \frac{(x+3)(x-1)}{(x+2)(x-1)}$$

   The graph has a vertical asymptote at $x = -2$ found from the factor $(x + 2)$ in the denominator.

   The graph also has a hole at $x = 1$ found from the factor $(x - 1)$ which shows up BOTH in the numerator and in the denominator!

Find the Equations of Vertical Asymptotes (2 of 2)

2. If the numerator and denominator of a rational function have NO factors in common, we say that it is reduced to lowest terms.

In this case, we set the entire denominator equal to 0 and solve. The solutions are then used to form equations of vertical lines which are the vertical asymptotes.

For example, the following function is reduced to lowest terms:

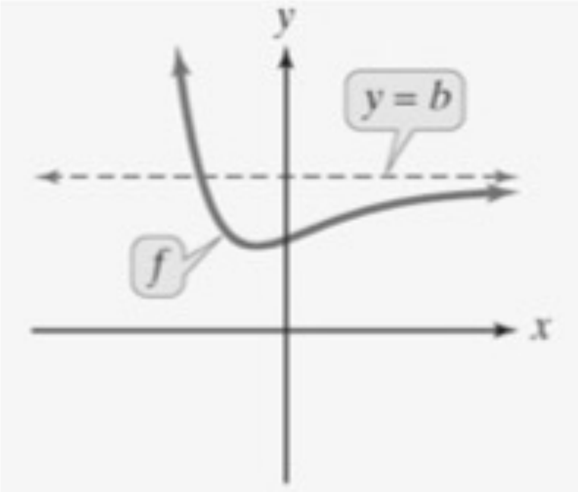
$$f(x) = \frac{x+3}{(x+2)(x-1)}$$

The numerator and the denominator have no factors in common! This means that the graph of the function has vertical asymptotes at $x = 1$ and at $x = -2$ (we set the factors in the denominator equal to 0!).

4. Definition of Horizontal Asymptotes

The graphs of some rational functions may also contain <u>at most one</u> horizontal asymptote, which is an invisible horizontal line. Some graphs may not have one!

The dashed line $y = b$ in the picture of a rational function below indicates a horizontal asymptote of the graph of the function f. **Some graphs might touch or cross their horizontal asymptote, however, there will always be an area where the graph will just continue to move toward it.**



When we graph rational functions by hand, the horizontal asymptote is drawn as a dashed line.
This is supposed to indicate that it is not a part of the graph.
When we graph rational functions with a graphing utility, the horizontal asymptote does not show up on the graph.

5. Find the Equations of Horizontal Asymptotes (1 of 2)

We find the equations of horizontal asymptotes by applying some calculus concepts which leads to the following three theorems.

Given a rational function $f(x) = \frac{p(x)}{q(x)}$, assume that

- the degree of the polynomial p is $\boldsymbol{n}$ and the degree of the polynomial q is $\boldsymbol{m}$.
- the leading coefficient of p is $\boldsymbol{a_n}$ and the leading coefficient of q is $\boldsymbol{b_m}$.

1. If $n < m$, the line $y = 0$ (x-axis) is the horizontal asymptote of the graph of f.

   Example: Given $k(x) = \frac{x^2+1}{4x^3-4x+9}$, we can see that $n = 2$ and $m = 3$.

   The line $y = 0$ (x-axis) is the horizontal asymptote of the graph of k.

Find the Equations of Horizontal Asymptotes (2 of 2)

2. If $n = m$, the line $y = \frac{a_n}{b_m}$ is the horizontal asymptote of the graph of *f*.

   Example: Given $h(x) = \frac{x-1}{4x}$, we can see that $n = 1$ and $m = 1$.

   The leading coefficient in the numerator is 1 and the one in the denominator is 4.

   The line $y = \frac{1}{4}$ is the horizontal asymptote of the graph of *h*.

3. If $n > m$, the graph of *f* has NO horizontal asymptote.

   Example: Given $p(x) = \frac{4x^3}{x^2+9}$, we can see that $n = 3$ and $m = 2$.

   The graph of *p* has NO horizontal asymptote.