Concepts
Polynomial Functions

Based on power point presentations by Pearson Education, Inc.

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Learning Objectives

1. Define polynomial functions.
2. Memorize the characteristics of graphs of polynomial functions.
3. Memorize and apply the characteristics of the *Zeros* of polynomial functions.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

1. Definition of a Polynomial Function (1 of 6)

We have already been exposed to many different functions. In this lesson, we will discuss yet another function, namely the *polynomial function*.

The **general form** of the polynomial function in *x* is

$$\boldsymbol{f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0}, \text{ where}$$

$a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are the coefficients which are real numbers with $a_n \neq 0$.

$x^n, x^{n-1}, \dots, x^2, x^1, x^0 = 1$ are their associated variables. Their exponents are strictly non-negative integers (positive integers and 0).

Domain: *All Real Numbers* or $(-\infty, \infty)$ in *Interval Notation*.

Definition of a Polynomial Function (2 of 6)

Examples of polynomial functions:

$g(x) = x^4 + 3x + 5$ (degree 4 and leading coefficient 1)

$p(x) = 3x^5 + (-4x) + (-9)$ (degree 5 and leading coefficient 3)

This function is usually written as $p(x) = 3x^5 - 4x - 9$. We eliminate the double signs!

$k(x) = -7x^{11} + 21$ (degree 11 and leading coefficient – 7)

$h(x) = -x^9$ (degree 9 and leading coefficient – 1)

Please note, in mathematics terms with the same variable are ALWAYS displayed in descending order of their exponent!

Definition of a Polynomial Expression (3 of 6)

More examples of polynomial functions:

$s(x) = 2x^3 - 5$ (degree 3 and leading coefficient 2, better known as a transformation of the cubic function $y = x^3$)

$q(x) = 6x^2 + 1$ (degree 2 and leading coefficient 6, better known as a quadratic function or as a transformation of the square function $y = x^2$)

$f(x) = 5x + 6$ (degree 1 and leading coefficient *5,* better known as a linear function)

$t(x) = 9$ (degree 0 and NO leading coefficient, better known as a constant function)

Definition of a Polynomial Expression (4 of 6)

The following are NOT polynomial functions:

$m(x) = 4x^3 + 3x^2 + 5x^{-1}$ is NOT a polynomial function because the exponents – 1 is not a positive integer.

$n(x) = 3x^5 + 5x^{\frac{2}{3}}$ is NOT a polynomial function because the exponent $\frac{2}{3}$ is not an integer.

Definition of a Polynomial Function (5 of 6)

Some Vocabulary:

Leading Coefficient - The leading coefficient of a polynomial function is the coefficient of the term containing the largest exponent on the variable.

For example, in $f(x) = -17x^3 + 4x^2 - 11x - 5$ the **leading coefficient is – 17**.

Degree - The degree of a polynomial function is equal to the largest exponent.

For example, in $f(x) = -17x^3 + 4x^2 - 11x - 5$ **the degree is 3**.

Definition of a Polynomial Function (6 of 6)

Monomial – A special name for a polynomial containing one term.

For example, 4 or x^2 or $7x^3$.

Binomial – A special name for a polynomial containing two terms.

For example, $4x^2 - 1$ or $x + 5$.

Trinomial – A special name for a polynomial containing three terms.

For example, $4x^2 - x - 5$.

2. Graphs of Polynomial Functions (1 of 7)

In this course, we will not graph polynomial functions by hand. However, we must memorize certain features of their graphs.

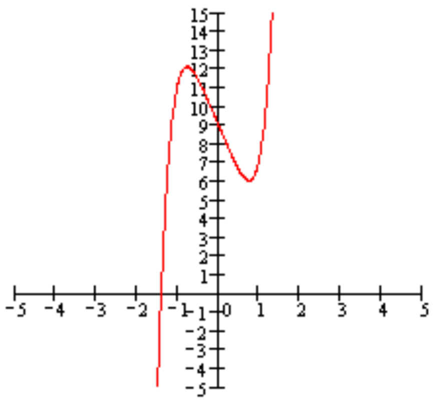
a. Polynomial functions have graphs that are smooth and continuous. That is, they have no sharp corners and no breaks.

b. Polynomial functions do not have one standard graph. Infinitely many different graphs are possible.

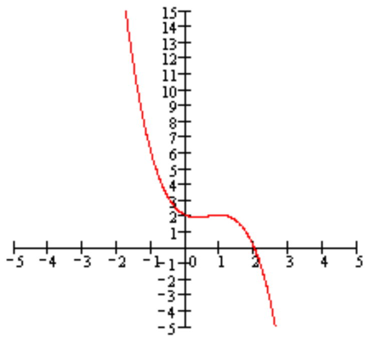
Graphs of Polynomial Functions (2 of 7)

Graphing utility drawn examples of graphs of polynomial functions.

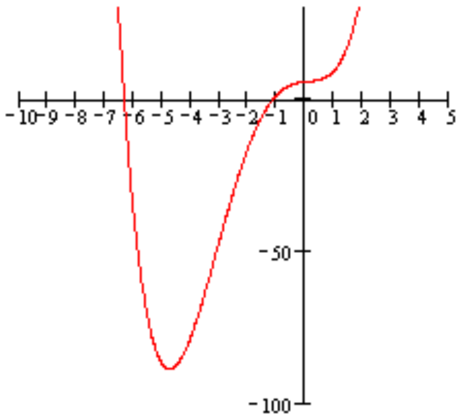
a.



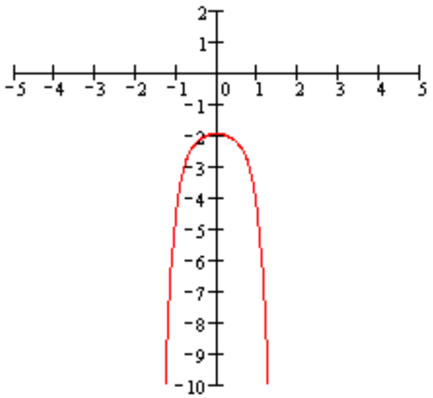
b.



c.



d.



Graphs of Polynomial Functions (3 of 7)

c. All graphs of polynomial functions have a certain **end behavior**. This is the behavior of the graphs as we let the *x*-values either get bigger and bigger or smaller and smaller.

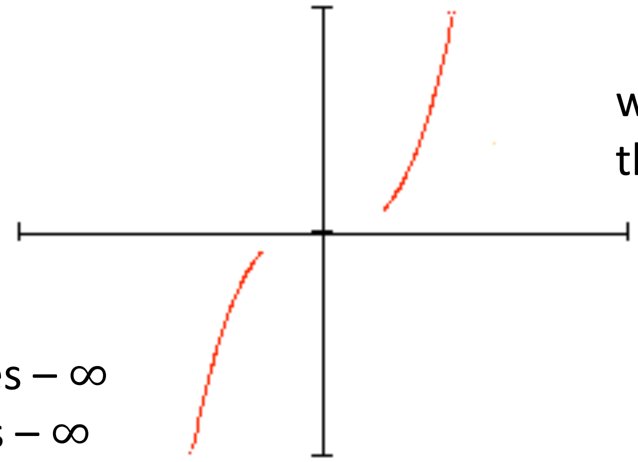
In mathematics, instead of saying ”bigger and bigger” we say, “*x* approaches positive infinity or $x \to \infty$”. Instead of saying “smaller and smaller” we say, “*x* approaches negative infinity or $x \to -\infty$”.

The degree and the leading coefficient of a polynomial function determine the end behavior.

Please examine the end behavior of the graphs in the previous slide.

Graphs of Polynomial Functions (4 of 7)

1) When the **degree of the polynomial is odd** and the **leading coefficient is positive**, the end behavior of the graph is as follows.



when x approaches $+\infty$
then y approaches $+\infty$

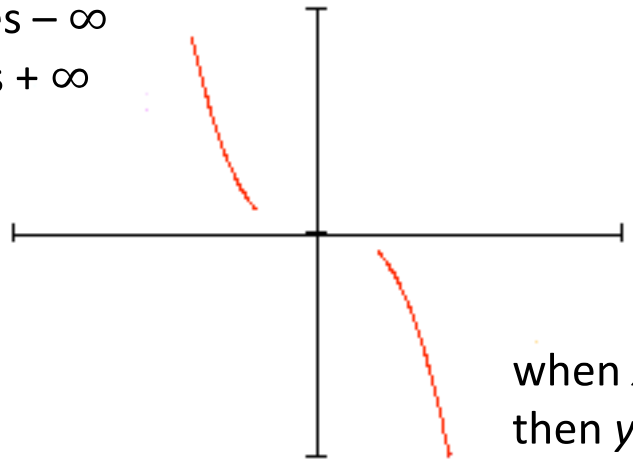
when x approaches $-\infty$
then y approaches $-\infty$

For an example see graph (a) on Slide 9!

Graphs of Polynomial Functions (5 of 7)

2) When the **degree of the polynomial is odd** and the **leading coefficient is negative**, the end behavior of the graph is as follows.

when *x* approaches – ∞
then *y* approaches + ∞



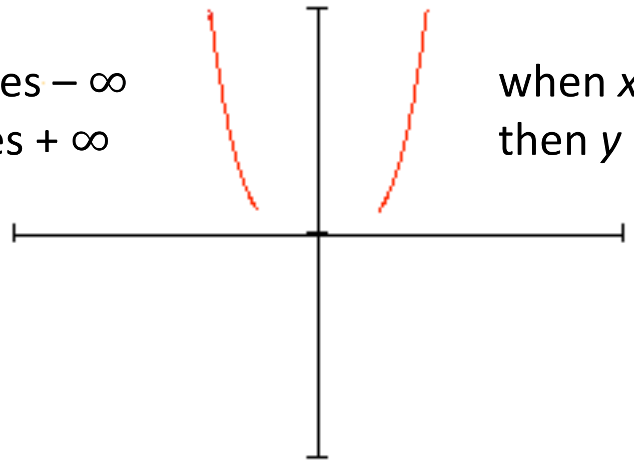
when *x* approaches + ∞
then *y* approaches – ∞

For an example see graph (b) on Slide 9!

Graphs of Polynomial Functions (6 of 7)

3) When the **degree of the polynomial is even** and the **leading coefficient is positive**, the end behavior of the graph is as follows.

when x approaches $-\infty$
then y approaches $+\infty$



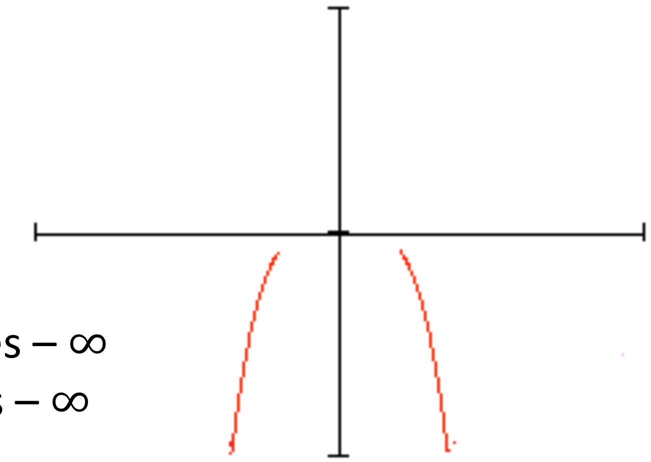
when x approaches $+\infty$
then y approaches $+\infty$

For an example see graph (c) on Slide 9!

Graphs of Polynomial Functions 7 of 7)

4) When the **degree of the polynomial is even** and the **leading coefficient is negative**, the end behavior of the graph is as follows.

when x approaches $-\infty$
then y approaches $-\infty$



when x approaches $+\infty$
then y approaches $-\infty$

For an example see graph (d) on Slide 9!

4. Zeros of Polynomial Functions (1 of 4)

If P is a polynomial function, then the values of x for which $P(x) = y = 0$ are called the *Zeros* of P.

For example, the function $P(x) = x^2 + x - 6$ has two *Zeros*, namely 2 and -3 because $P(2) = 0$ and $P(-3) = 0$. In both cases, the y-variable equals 0!

a. The degree of the polynomial function tells us how many *Zeros* we will get (*Fundamental Theorem of Algebra*)

For example, the degree of $P(x) = 8x^3 - 5x^2 + 3x + 9$ is 3, therefore, there will be 3 *Zeros*.

b. *Zeros* can be real or imaginary.

Zeros of Polynomial Functions (2 of 5)

c. Imaginary *Zeros* appear in conjugate pairs.

For example, if $-3 + 2i$ is a *Zero* of a polynomial function, then $-3 - 2i$ must also be a *Zero*. Likewise, if $2i$ is a *Zero*, then $-2i$ must also be a *Zero*.

d. *Zeros* do not have to be distinct.

For example, we are told that $-1, 4, 4$ are the *Zeros* of a polynomial function of degree 3. Notice that the 4 shows up twice! That's when we say that the *Zero* 4 is NOT distinct.

Zeros of Polynomial Functions (3 of 5)

e. If a number ***r*** is a *Zero* of a polynomial function in x, then **(x – r)** is a **linear factor**** of the function and vice versa. This statement has a name. It is called the *Factor Theorem*.

   For example, assume a polynomial function P of degree 3 has *Zeros* – 1 and 2. By the *Factor Theorem* the following are linear factors of P:

   $(x - (-1)) = (x + 1)$

   $(x - (+2)) = (x - 2)$

****** A **linear factor** only has a variable term raised to the first power!

Zeros of Polynomial Functions (5 of 5)

f. If a number ***r*** is a *Zero* of a polynomial function in x and $\boldsymbol{(x - r)^m}$ is a factor of a polynomial function, then ***r*** is called a *Zero* of multiplicity m.

For example, in the polynomial function $P(x) = (x - 5)^2(x - 7)^3$, 5 is a *Zero* of multiplicity 2 and 7 is a *Zero* of multiplicity 3.

Please note that the polynomial function is not written in general form. Instead, it is written as a product of linear factors! If we multiply and combine like terms we eventually get back to standard form.