Concepts
Polynomial Functions

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Define polynomial expressions.
2. Define polynomial functions.
3. Memorize the characteristics of graphs of polynomial functions.
4. Memorize and apply the characteristics of the *Zeros* of polynomial functions.

1. Definition of a Polynomial Expression (1 of 4)

In the past, we mentioned and worked with mathematical expressions. We said that they are a combination of variables and numbers linked by addition, subtraction, multiplication, and division.

A special type of expression is called a polynomial expression or polynomial for short, and it has certain characteristics. The polynomial expression in *x* is written in standard form as follows:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where a_n , a_{n-1} , … a_2 , a_1 , a_0 are the coefficients which are real numbers with $a_n \neq 0$.

x^n , x^{n-1}, … , x^2, x are their associated variables. Their exponents are strictly non-negative integers (positive integers and 0).

Definition of a Polynomial Expression (2 of 4)

Examples of Polynomial Expressions:

$5x$

$x - 4$

$x^2 - 5x + 10$

$6x^7 + 2x^5 + 11x^4 - x^3 + 3x^2 - x + 4$

Please note, in mathematics terms with the same variable are ALWAYS displayed in descending order of their exponent!

Note the following:

$4x^3 - 3x^2 + 5x^{-1}$ is NOT a polynomial expression because the exponents -1 is not a positive integer.

Likewise, $3x^5 - 5x^{\frac{2}{3}}$ is NOT a polynomial expression because the exponent $\frac{2}{3}$ is not an integer.

Definition of a Polynomial Expression (3 of 4)

Some Vocabulary:

Leading Coefficient - The leading coefficient of a polynomial expression is the coefficient of the term containing the largest exponent on the variable.

For example, in $-17x^3 + 4x^2 - 11x - 5$ the **leading coefficient is – 17**.

Degree - The degree of a polynomial expression is equal to the largest exponent .

For example, in $-17x^3 + 4x^2 - 11x - 5$ **the degree is 3**.

Definition of a Polynomial Expression (4 of 4)

Monomial – Polynomials containing one term, for example $5x$, are also called monomials.

Binomial - Polynomials containing two terms, for example $x – 4$, are also called binomials.

Trinomial - Polynomials containing three terms, for example $x^2 – 5x + 10$, are also called trinomials.

2. Definition of a Polynomial Function

As soon as we add a second variable to the polynomial expression, we get a polynomial function. Mathematically, a polynomial function in *x* is usually written in standard form as follows:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where f(*x*) = *y*

Domain: *All Real Numbers* or $(-\infty, \infty)$ in *Interval Notation*.

Examples:

$f(x) = x^4 + 3x - 5$ A polynomial function of **degree** 4 with a **leading coefficient** of 1.

$f(x) = 3x^5 - 5x + 9$ A polynomial function of **degree** 5 with a **leading coefficient** of 3.

$g(x) = 4x^3 - 3x^2 + 5x^{-1}$ NOT a polynomial function because the exponents – 1 is NOT a positive integer.

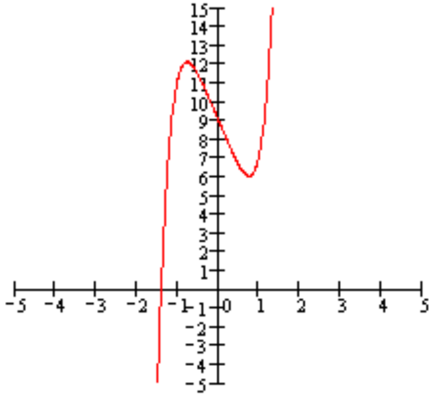
3. Characteristics of Graphs of Polynomial Functions (Memorize!) (1 of 7)

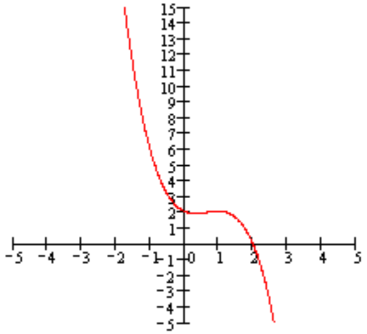
In this course, we will not graph polynomial functions by hand. However, we must memorize certain features of their graphs.

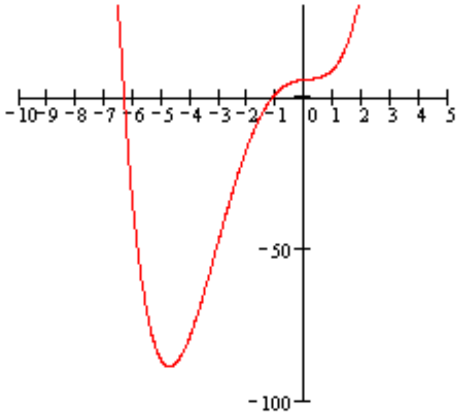
1. Polynomial functions have graphs that are smooth and continuous. That is, they have no sharp corners and no breaks.
2. Polynomial functions do not have one standard graph. Infinitely many different graphs are possible.

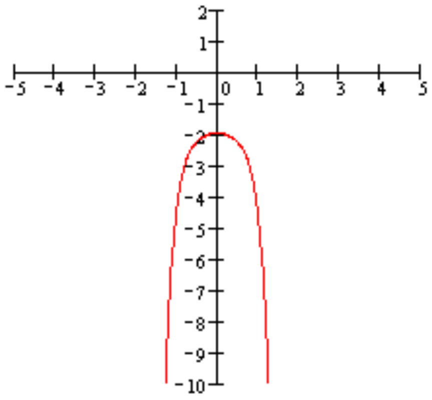
Characteristics of Graphs of Polynomial Functions (Memorize!) (2 of 7)

Examples of graphs of polynomial functions.









Characteristics of Graphs of Polynomial Functions (Memorize!) (3 of 7)

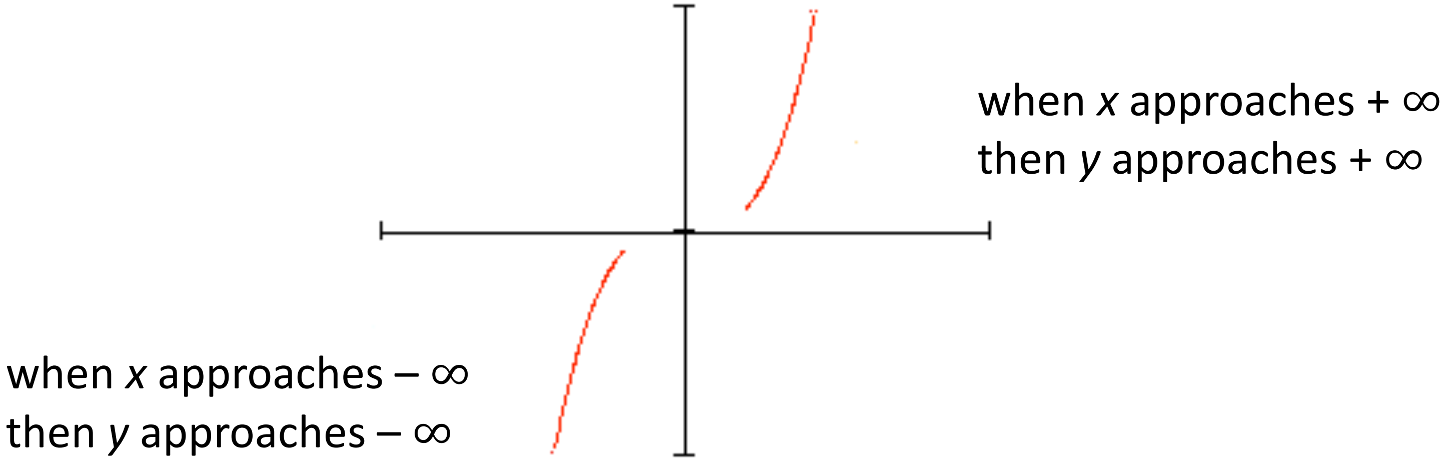
3. All graphs of polynomial functions have a certain end behavior. This is the behavior of the graphs as we let the *x*-values either get bigger and bigger or smaller and smaller. In mathematics we say, “*x* approaches positive infinity ($x \to \infty$)” or “*x* approaches negative infinity ($x \to -\infty$)”.

The degree and the leading coefficient of a polynomial function determine the end behavior.

Please examine the end behavior of the graphs in the previous slide.

Characteristics of Graphs of Polynomial Functions (Memorize!) (4 of 7)

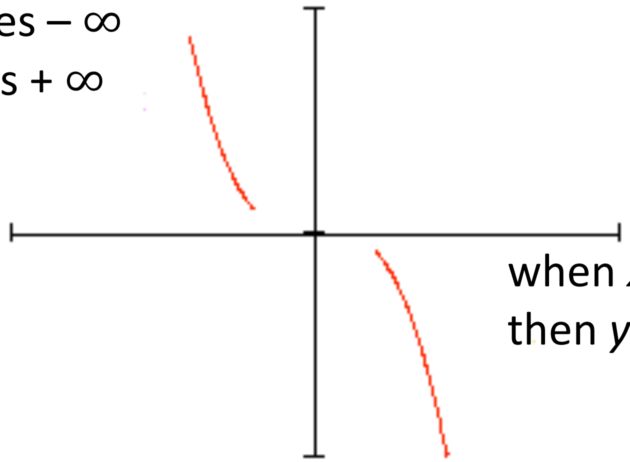
a. When the **degree of the polynomial is odd** and the **leading coefficient is positive**, the end behavior of the graph is as follows.



Characteristics of Graphs of Polynomial Functions (Memorize!) (5 of 7)

b. When the **degree of the polynomial is odd** and the **leading coefficient is negative**, the end behavior of the graph is as follows.

when x approaches $-\infty$
then y approaches $+\infty$

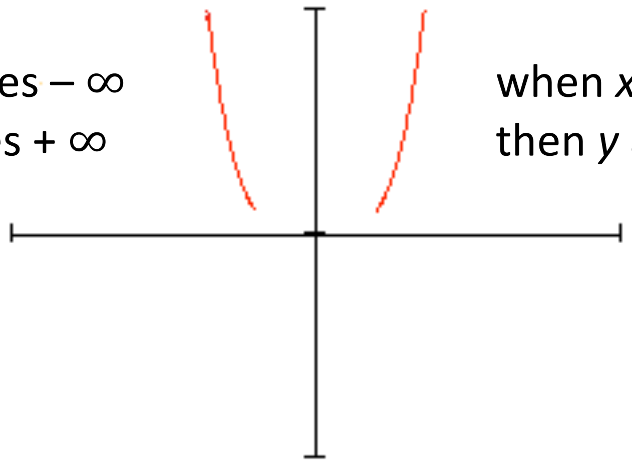


when x approaches $+\infty$
then y approaches $-\infty$

Characteristics of Graphs of Polynomial Functions (Memorize!) (6 of 7)

c. When the **degree of the polynomial is even** and the **leading coefficient is positive**, the end behavior of the graph is as follows.

when x approaches – ∞
then y approaches + ∞

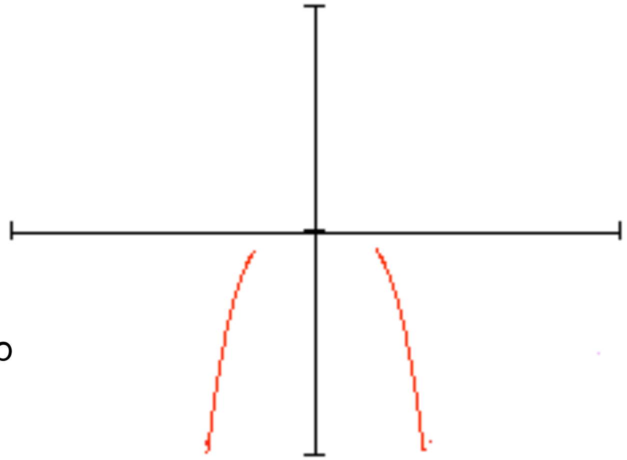


when x approaches + ∞
then y approaches + ∞

Characteristics of Graphs of Polynomial Functions (Memorize!) (7 of 7)

d. When the **degree of the polynomial is even** and the **leading coefficient is negative**, the end behavior of the graph is as follows.

when x approaches $-\infty$
then y approaches $-\infty$



when x approaches $+\infty$
then y approaches $-\infty$

4. Zeros of of Polynomial Functions and their Characteristics (Memorize!) (1 of 5)

If P is a polynomial function, then the values of x for which $P(x) = y = 0$ are called the *Zeros* of P.

For example, the polynomial function $P(x) = x^2 + x – 6$ of degree 2 (also called quadratic function) has two *Zeros*, namely 2 and – 3.

Please observe $P(2) = (2)^2 + 2 – 6 = 0$ and $P(–3) = (–3)^2 + (–3) – 6 = 0$.

Zeros of of Polynomial Functions and their Characteristics (Memorize!) (2 of 5)

a. The degree of the polynomial function tells us how many *Zeros* we will get (*Fundamental Theorem of Algebra*)

   For example, the degree of $P(x) = 8x^3 - 5x^2 + 3x + 9$ is 3, therefore, we will have 3 *Zeros*.

b. *Zeros* do not have to be distinct.

   For example, we are told that –1, 4, 4 are the *Zeros* of a polynomial function of degree 3. Notice that the 4 shows up twice! That's when we say that the *Zero* 4 is NOT distinct.

c. *Zeros* can be real or imaginary.

Zeros of of Polynomial Functions and their Characteristics (Memorize!) (3 of 5)

d. Imaginary *Zeros* appear in conjugate pairs.

For example, if – 3 + 2i is a *Zero* of a polynomial function, then – 3 – 2i must also be a *Zero*. Likewise, if 2i is a *Zero*, then – 2i must also be a *Zero*.

Note: The conjugate of a complex number $a + bi$ is the complex number $a – bi$!

Zeros of of Polynomial Functions and their Characteristics (Memorize!) (4 of 5)

e. If a number ***r*** is a *Zero* of a polynomial function, then **(*x* – *r*)** is a **linear factor** of the function and vice versa. This statement has a name. It is called the *Factor Theorem*.

For example, assume a polynomial function *P* of degree 3 has *Zeros* –1, 1, 2. By the *Factor Theorem,*

$$\mathbf{(x - (-1)) = (x + 1)}$$
$$\mathbf{(x - (+1)) = (x - 1)}$$
$$\mathbf{(x - (+2)) = (x - 2)}$$

are **linear factors** of the function.

As a matter of fact, we can now find the equation of the function which is $P(x) = (x + 1)(x - 1)(x - 2)$.

Zeros of of Polynomial Functions and their Characteristics (Memorize!) (5 of 5)

f. If $(\boldsymbol{x} - \boldsymbol{r})^m$ is a factor of a polynomial function, then $\boldsymbol{r}$ is called a *Zero* of multiplicity m (the power of the factor).

For example, we are told that –1, 3, 3, 5, 5, 5, 5 are the *Zeros* of a polynomial function of degree 7. Please note that 3 and 5 are not distinct.

In this case, we say

Zero 3 has multiplicity 2 and
Zero 5 has multiplicity 4 and
Zero – 1 has multiplicity 1