Concepts
Solve Systems of Linear Equations Using Matrices

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Learning Objectives

1. Define a matrix.
2. Define augmented matrices and write one for a system of linear equations.
3. Transform matrix rows.
4. Solve systems of linear equations using the matrix method.

1. Definition of a Matrix (1 of 4)

In this lesson, we will learn another method for solving systems of linear equations. It relies heavily on something called a matrix (plural: matrices).

A matrix is an array of numbers, arranged in rows and columns and placed in brackets. Sometimes, we give matrices names usually using capital letters, such as *A*, *B*, or *C*. Each number in a matric is called an **element**.

For example, $C = \begin{bmatrix} 5 & -2 & 8 \\ -3 & 4 & 1 \\ 1 & 6 & 0 \end{bmatrix}$.

Definition of a Matrix (2 of 4)

Every matrix is said to be of a certain order which is usually expressed as $\boldsymbol{m \times n}$ (*m by n*). The ***m*** indicates the number of rows, and the ***n*** indicates the number of columns. The row number is always stated first.

For example, $A = \begin{bmatrix} 5 & -2 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$ 3 × 2 matrix

or $B = \begin{bmatrix} 5 & -2 \\ -3 & 4 \end{bmatrix}$ 2 × 2 matrix, also called a square matrix because the number of rows equal the number of columns.

Definition of a Matrix (3 of 4)

Matrices must be of the same order to perform addition and subtraction. Each element in the corresponding position is then added or subtracted.

Example 1:

Perform the indicated matrix operations: $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

Definition of a Matrix (4 of 4)

A **scalar** in matrix mathematics is a single real number. When a matrix is multiplied by a scalar, we call this **scalar multiplication**.

Then, given a matrix *A* and a scalar *c*, the matrix *cA* is obtained by multiplying each element of *A* by the real number *c*.

Example 2:

If $B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$, find the matrix – 6*B.*

Solution:

$$-6B = -6\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} (-6)(-1) & (-6)(-2) \\ (-6)(8) & (-6)(5) \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix}$$

2. Definition of Augmented Matrices (1 of 2)

We will now add a vertical bar to separate the rightmost column of the matrix from the other columns. We call this an augmented matrix.

For example, $\left[\begin{array}{cc|c} 4 & 12 & 8 \\ -3 & 6 & 7 \end{array}\right]$

or $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 7 & 6 & -3 & 7 \\ -3 & -2 & 4 & -9 \end{array}\right]$

Definition of Augmented Matrixes (2 of 2)

Row-Echelon Form of an Augmented Matrix

An augmented matrix is said to be in row-echelon form if the following occurs on the left side of the vertical bar:

- Ones are shown on the main diagonal which runs from the top left corner of the matrix to the bottom right corner.
- Zeros are above and below the ones.

$$\left[\begin{array}{cc|c} 1 & 0 & A \\ 0 & 1 & B \end{array}\right]$$

A and *B* are any real numbers.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right]$$

A, *B*, and *C are* any real numbers.

3. Allowable Row Operations (1 of 5)

Any row in a matrix can be transformed by performing one or more of the following three allowable row operations:

1. All elements in any row R of a matrix may be multiplied by some nonzero number, let's call it k. Symbolically, this can be expressed as kR_i where i is the position of the rows in the matrix.

Example 3:

Given the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ multiply all elements in Row 1 by $\frac{1}{4}$.

Allowable Row Operations

Example 3 continued:

We will write the instructions as $\frac{1}{4}R_1$ next to Row 1.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} \frac{1}{4}(4) & \frac{1}{4}(12) & \frac{1}{4}(-20) & \frac{1}{4}(8) \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

Allowable Row Operations (3 of 5)

2. Two rows R in a matrix can be added, and their sum can replace any row. Symbolically, this can be expressed as $kR_i + R_j$ where i and j is the position of the rows in the matrix and k is some nonzero number.

Example 4:

Given the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$

replace Row 3 with the following sum: $3R_2 + R_3$

This means to multiply every element in Row 2 by 3 then add to this product the "corresponding" elements in Row 3 (elements in the same column!). We will write these instructions next to Row 3 because that's the one we are replacing.

Allowable Row Operations (4 of 5)

Example 4 continued:

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \; 3R_2 + R_3 \qquad = \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ \underset{3\,R2}{3(1)}+\underset{R3}{(-3)} & \underset{3\,R2}{3(6)}+\underset{R3}{(-2)} & \underset{3\,R2}{3(-3)}+\underset{R3}{1} & \underset{3\,R2}{3(7)}+\underset{R3}{(-9)} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 0 & 16 & -8 & 12 \end{array}\right]$$

Note that Row 3 was changed from [– 3 – 2 1 | – 9] to [0 16 – 8 | 12].

Allowable Row Operations (5 of 5)

3. Two rows R of a matrix may be interchanged. Symbolically, this can be expressed as $R_i \leftrightarrow R_j$ where i and j is the position of the rows in the matrix.

   Example 5:

   Given the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ interchange Row 1 and Row 2.

   We will write the instructions as $R_1 \leftrightarrow R_2$ in between Row 1 and Row 2.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] R_1 \leftrightarrow R_2 \quad = \left[\begin{array}{ccc|c} 1 & 6 & -3 & 7 \\ 4 & 12 & -20 & 8 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

4. Solve Systems of Linear Equations Using the Matrix Method (1 of 5)

In earlier lectures, we solved systems of linear equations using the Substitution and/or Addition Methods. Now we are going to the Matrix Method to solve systems of linear equations.

This method is also called the *Gauss-Jordan Elimination Method*. It was created by the German mathematician Carl Friedrich Gauss (1777-1855) and the German geodesist (someone who measures and monitors the earth) Wilhelm Jordan (1842-1899).

Solve Systems of Linear Equations Using the Matrix Method

(2 of 5)

Step 1 - Place the equations in a system into an augmented matrix using only the coefficients of the variables. Like coefficients of like variables appear in the same column.

NOTE: In mathematics, we use alphabetic order to write columns. For example, the x-column is first, then the y-column, then the z-column, and finally the constant column.

Solve Systems of Linear Equations Using the Matrix Method
(3 of 5)

Example 6:

Solve the, the system $\begin{cases} 2x+y=1 \\ 3x+2y=4 \end{cases}$ using the Matrix Method.

Let's write the system into an augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right]$$

Note that the first column is the *x*-column and the second column the *y*-column. The vertical bar is the equal sign, and the last column consist of the constants.

Solve Systems of Linear Equations Using the Matrix Method

Step 2 - Change the augmented matrix to row-echelon form **using the three allowable row operations discussed earlier**. The row operations MUST be completed in the order as indicated by the subscripts on the elements below!

$$\left[\begin{array}{cc|c} 1_1 & 0_4 & A \\ 0_2 & 1_3 & B \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 1_1 & 0_9 & 0_8 & A \\ 0_2 & 1_4 & 0_7 & B \\ 0_3 & 0_5 & 1_6 & C \end{array}\right]$$

NOTE: The first allowable row operation allows us to transform elements to 1. The second allowable row operation allows us to transform elements to 0.

Solve Systems of Linear Equations Using the Matrix Method
(5 of 5)

Step 3 - Once the augmented matrix is in row-echelon form it becomes the **solution matrix**. It allows us to identify the solution of the system by looking at the 1's!

Example 6 continued:

We will soon find out that the **solution matrix** for the system $\begin{cases} 2x + y = 1 \\ 3x + 2y = 4 \end{cases}$ is

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$$

We can then state that ***x* = – 2** and ***y* = 5**. Graphically this means that the two lines, defined by the two equations in the system, intersect at the point created by the ordered pair (– 2 , 5).