Concepts
Solve Systems of Linear Equations Using Matrices

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Learning Objectives

1. Define augmented matrices and write one for a system of linear equations.
2. Transform augmented matrices to row-echelon form.
3. Solve systems of linear equations using the matrix method.

1. Definition of Augmented Matrices (1 of 2)

An augmented matrix has a vertical bar separating the columns of the matrix into two groups.

Examples:

$$\left[\begin{array}{cc|c} 4 & 12 & 8 \\ -3 & 6 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 7 & 6 & -3 & 7 \\ -3 & -2 & 4 & -9 \end{array}\right]$$

Definition of Augmented Matrixes (2 of 2)

Row-Echelon Form of an Augmented Matrix

An augmented matrix is said to be in row-echelon form if the following occurs on the left side of the vertical bar:

- Ones are shown on the main diagonal which runs from the top left corner of the matrix to the bottom right corner.
- Zeros are above and below the ones.

$$\left[\begin{array}{cc|c} 1 & 0 & A \\ 0 & 1 & B \end{array}\right]$$

A and *B* are any real numbers.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right]$$

A, *B*, and *C are* any real numbers.

2. Transform Augmented Matrices to Row-Echelon Form

Any augmented matrix can be changed to row-echelon form by performing the following three “allowable” row operations:

1. Two rows R of an augmented matrix may be interchanged. Symbolically, this can be expressed as $R_i \leftrightarrow R_j$ where i and j is the order of the rows. This is used, when appropriate, to produce 1’s or 0’s in the row echelon form.

Example:

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and interchange Row 1 and Row 2.

Transform Augmented Matrices to Row-Echelon Form

Interchange Row 1 and Row 2. We will write these instructions as $R_1 \leftrightarrow R_2$ in between Row 1 and Row 2.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \begin{array}{c} \\ R_1 \leftrightarrow R_2 \\ \\ \end{array} = \left[\begin{array}{ccc|c} 1 & 6 & -3 & 7 \\ 4 & 12 & -20 & 8 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

Notation we will use:

Row 1: R_1
Row 2: R_2
Row 3: R_3

Transform Augmented Matrices to Row-Echelon Form

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2. All elements in any row R may be multiplied by some nonzero number, let's call it k. Symbolically, this can be expressed as kR_i where i is the order of the row. This produces 1's in the row echelon form.

Example:

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and change the first element in Row 1 from 4 to 1.

Multiply all elements in Row 1 by $\frac{1}{4}$. We will write the instructions as $\frac{1}{4}R_1$ next to Row 1.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array} = \left[\begin{array}{ccc|c} \frac{1}{4}(4) & \frac{1}{4}(12) & \frac{1}{4}(-20) & \frac{1}{4}(8) \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

Transform Augmented Matrices to Row-Echelon Form

3. Two rows R can be added to replace a row. Symbolically, this can be expressed as $kR_i + R_j$ where i and j is the order of the rows and k is some nonzero number. This produces 0's in the row echelon form.

Example:

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and change the first element in Row 3 from – 3 to 0.

Replace Row 3 with the following sum: $3R_2 + R_3$

This means to multiply every element in Row 2 by 3 then add this product to the corresponding elements in Row 3. We will write these instructions next to Row 3.

Transform Augmented Matrices to Row-Echelon Form

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \; 3R_2 + R_3 \quad = \left[\begin{array}{cccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ \underbrace{3(1)}_{3\,R2}+\underbrace{(-3)}_{R3} & \underbrace{3(6)}_{3\,R2}+\underbrace{(-2)}_{R3} & \underbrace{3(-3)}_{3\,R2}+\underbrace{1}_{R3} & \underbrace{3(7)}_{3\,R2}+\underbrace{(-9)}_{R3} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 0 & 16 & -8 & 12 \end{array}\right]$$

Note that Row 3 was changed from [– 3 – 2 1 **|** – 9] to [0 16 – 8 **|** 12].

Transform Augmented Matrices to Row-Echelon Form

Below are some very helpful hints to help us write the "scary" calculation in the third allowable operation. Let's look at the following matrix and its row operation.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \quad 3R_2 + R_3$$

This calculation changes the first element in Row 3 from – 3 to 0.

1. **The row we add in the calculation is always the one that we replace.** For example, in $3R_2 + R_3$ we add R_3 because we are replacing Row 3.
2. **The nonzero coefficient we use in the calculation is always the negative of the number we want to change into a 0.** For example, in $3R_2 + R_3$ we use 3 because the number we want to change to a 0 is – 3 (first element in Row 3).

3. Solve Systems of Linear Equations Using the Matrix Method (1 of 3)

In earlier lectures, we solved systems of linear equations using the Substitution and Addition Methods. Now we are going to the Matrix Method to solve systems of linear equations.

This method is also called the *Gauss-Jordan Elimination Method*. It was created by the German mathematician Carl Friedrich Gauss (1777-1855) and the German geodesist (someone who measures and monitors the earth) Wilhelm Jordan (1842-1899).

Solve Systems of Linear Equations Using the Matrix Method
(2 of 3)

1. Place the equations in a system into an augmented matrix using only the coefficients of the variables. Like coefficients of like variables appear in the same column.

   Example:

   The system $\begin{cases} 2x+y=1 \\ 3x+2y=4 \end{cases}$ can be change to augmented matrix $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & 2 & 4 \end{array}\right]$.

2. Change the augmented matrix to row-echelon form using the three row operations discussed earlier. The row operations MUST be completed in the order as indicated by the subscripts on the elements below!

$$\left[\begin{array}{cc|c} 1_1 & 0_4 & A \\ 0_2 & 1_3 & B \end{array}\right] \quad \text{OR} \quad \left[\begin{array}{ccc|c} 1_1 & 0_9 & 0_8 & A \\ 0_2 & 1_4 & 0_7 & B \\ 0_3 & 0_5 & 1_6 & C \end{array}\right]$$

Solve Systems of Linear Equations Using the Matrix Method

Example continued:

We will soon find out that its **solution matrix** is $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \end{array}\right]$.

3. Once you have a **solution matrix** you can identify the solutions of the variables by looking at the 1's!

Example continued:

We can then state the solutions for ***x*** and ***y***, namely ***x = – 2*** and ***y = 5***. Graphically this means that the two lines, defined by the two equations in the system, intersect at the point ***(– 2 , 5)***.