Concepts
Linear Equations in One Variable

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Learning Objectives

1. Memorize the *Basic Principles of Equations* consisting of four axioms.
2. Memorize the definition of linear equations.
3. Solve linear equations containing integers.
4. Solve linear equations containing fractions.
5. Memorize and use the *Cross-Multiplication Principle*.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

1. Basic Principles of Equations (1 of 3)

We were told previously that equations consist of mathematical expressions equal to other mathematical expressions. We now continue our study of equations, and the first thing we will learn is that any equation can be transformed into an equivalent equation by using the **Basic Principles of Equations**.

They consist of the following four axioms (statements that are self-evident and require no proof).

1. **Addition Axiom** - Add the same quantity to **BOTH** sides of the equal sign.

   For example, given $x - 9 = 3 + 2x$, add 9 to both sides of the equal sign. We write $x - 9 + 9 = 3 + 2x + 9$

Basic Principles of Equations

2. **Subtraction Axiom** - Subtract the same quantity from **BOTH** sides of the equal sign.

   For example, given $4x + 5 = 1 + 3x$, subtract $3x$ from both sides of the equal sign. We write $4x + 5 - 3x = 1 + 3x - 3x$.

3. **Multiplication Axiom** – Multiply **BOTH** sides of the equal sign by the same **nonzero** quantity.

   For example, given $3x = 9$, multiply both sides of the equal sign by $\frac{1}{3}$.

   We write $\frac{1}{3}(3x) = \frac{1}{3}(9)$.

Basic Principles of Equations

4. **Division Axiom** – Divide **BOTH** sides of the equal sign by the same **nonzero** quantity.

   For example, given $2x = 8$, divide both sides of the equal sign by 2.

   We write $\frac{2x}{2} = \frac{8}{2}$.

   NOTE: A fraction bar indicates division ÷.

2. Linear Equations in One Variable (1 of 2)

In mathematics, we encounter many different equations with one, two, or three variables, mostly using *x*, *y*, and *z*. We usually define these equations in *general form* although in our work we often don't run into this form.

One important equation is the **linear equation** in one variable, say *x*. Its **general form** is as follows:

$\boldsymbol{Ax + B = 0}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are real numbers with $\boldsymbol{A \neq 0}$

Please note that linear equations in one variable do not necessarily have to appear in *general form*. Mathematics just likes to define them that way!

Linear Equations in One Variable (2 of 2)

Examples of linear equations in one variable:

$5x + 9 = 0$ (general form with $A = 5$ and $B = 9$)

Please note that $5x$ means $5 \cdot x$.

$3p = 0$ (general form with $A = 3$ and $B = 0$)

$-4x + (-6) = 0$ (general form with $A = -4$ and $B = -6$)

Please note that this equation is usually written as $-4x - 6 = 0$. We eliminate the double signs!

$-k - 7 = 4$ (not in general form, but still a linear equation in one variable)

3. Solve Linear Equations Containing Integers (1 of 6)

When we are asked to "solve" an equation, say in *x,* we are actually asked to determine all values of *x* that result in a true statement when substituted into the equation. Such values are called **solutions**.

IMPORTANT NOTE: When solving equations, the goal is always to "isolate" the variable. In mathematics, "to isolate" means that we want to end up with the variable by itself on one side of the equal sign … mathematicians prefer the left side … having a coefficient of 1. The other side then gives us the "solution."

We accomplish "isolation" by using any combination of the Addition, Subtraction, Multiplication, and Division Axioms. We might have to use the axioms more than once.

Solve Linear Equations Containing Integers (2 of 6)

Strategy for Solving Linear Equations Containing Integers:

Step 1: If necessary, simplify the expressions on each side of the equal sign (remove grouping symbols, combine like terms, etc.).

Example 1:

Solve the linear equation $4(2x - 1) = 3(2x - 5) + 12$.

Note, this equation is NOT in *general form*!

$8x - 4 = 6x - 15 + 12$ Removed the parentheses by applying the Distributive Property!

$8x - 4 = 6x - 3$ Combined like terms on the right side of the equal sign!

Solve Linear Equations Containing Integers

Step 2: If applicable, use the Addition and/or the Subtraction Axiom to get all variable terms to one side of the equal sign and all constants to the other side. Note, in mathematics we like to place the variable terms on the left side, although it does not make a difference in the outcome. Combine any like terms.

Example 1 continued with $8x - 4 = 6x - 3$:

Let's get all variables to the left side of the equal sign. Therefore, we will subtract $6x$ from both sides (Subtraction Axiom). Then we will combine any like terms.

$8x - 4 - 6x = 6x - 3 - 6x$

and $2x - 4 = -3$

Solve Linear Equations Containing Integers

Example 1 continued:

Given $2x - 4 = -3$, let's get all constants to the right side of the equal sign. Therefore, we will add 4 to both sides (Addition Axiom). Then we will combine any like terms.

$2x - 4 + 4 = -3 + 4$

and $2x = 1$

Solve Linear Equations Containing Integers (5 of 6)

Step 3: Once we have one variable term on one side of the equal sign and the constant term on the other side, we might have to further "isolate" the variable by using the Multiplication and Division Axioms. This means the coefficient of the variable MUST be +1 (never –1). This is the proposed solution to the equation.

Example 1 continued:

Given $2x = 1$, we will "isolate" the variable by dividing both sides of the equation by 2 (which is the coefficient of the variable). That is, we are using the Division Axiom!

$$\frac{2x}{2} = \frac{1}{2}$$

$$x = \frac{1}{2}$$ This is the proposed solution.

Solve Linear Equations Containing Integers (6 of 6)

Step 4: Replace the variable in the original equation with the proposed solution. If the result is a true statement, then the proposed solution is an actual solution.

Example 1 continued:

Given $x = \frac{1}{2}$, we replace x in the original equation $4(2x - 1) = 3(2x - 5) + 12$ with $\frac{1}{2}$.

$4\left(2\left(\frac{1}{2}\right) - 1\right) = 3\left(2\left(\frac{1}{2}\right) - 5\right) + 12$

$4(1 - 1) = 3(1 - 5) + 12$

$4(0) = 3(-4) + 12$

$4(0) = -12 + 12$

and $0 = 0$. This is a true statement, therefore the proposed solution $x = \frac{1}{2}$ is an actual solution.

4. Solve Linear Equations Containing Fractions (1 of 6)

When we encounter fractions in a linear equation, we usually want to quickly ”eliminate” them to make our job of solving the equation easier. Following is a strategy on how to do this.

Solving Linear Equations involving Fractions:

Step 1: Find a number divisible by all denominators. Although it makes no difference in the end, we usually prefer to find the smallest such number because the subsequent calculations may not be as cumbersome.

Example 2:

Solve the linear equation $\frac{x-3}{4}=\frac{5}{14}-\frac{x+5}{7}$

Solve Linear Equations Containing Fractions (2 of 6)

Example 2 continued:

Let's find a number divisible by all denominators in $\frac{x-3}{4} = \frac{5}{14} - \frac{x+5}{7}$. Please understand that we can always find such a number by calculating the product of all denominators. In our case, that would be 4(14)(7) = 392.

However, you might notice that 28 is also a number divisible by 4, 14, and 7. It certainly is a lot smaller than 392 and therefore easier to work with. Actually, it is the smallest number divisible by all denominators.

Step 2: Multiply both sides of the equation by the number found in Step 1 and solve. This is a proposed solution!

Example 2 continued:

Let's use 28, the smallest number divisible by all denominators.

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14} - \frac{x+5}{7}\right)$$

Solve Linear Equations Containing Fractions (3 of 6)

Example 2 continued:

Before we continue, let’s swap the negative sign of the second term on the right side of the equal sign with the positive sign of 28! This is acceptable and allows for easier calculations.

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14}\right) + 28\left(-\frac{x+5}{7}\right)$$

We end up with the following:

$$28\left(\frac{x-3}{4}\right) = 28\left(\frac{5}{14}\right) - 28\left(\frac{x+5}{7}\right)$$

and after some canceling, we get $7(x - 3) = 2(5) - 4(x + 5)$.

Solve Linear Equations Containing Fractions (4 of 6)

Example 2 continued:

Using the *Distributive Property*, we find $7x - 21 = 10 - 4x - 20$ and $7x - 21 = -4x - 10$.

Then $11x = 11$ and $x = 1$. This is the proposed solution.

Step 3: Replace the variable in the original equation with the proposed solution. If the result is a true statement, then the proposed solution is an actual solution.

Solve Linear Equations Containing Fractions

Example 2 continued:

Let's replace *x* in the original equation $\frac{x-3}{4} = \frac{5}{14} - \frac{x+5}{7}$ with 1.

$$\frac{1-3}{4} = \frac{5}{14} - \frac{1+5}{7}$$

$$\frac{-2}{4} = \frac{5}{14} - \frac{6}{7}$$

NOTE: When we add or subtract fractions with "unlike" denominators, we must change all denominators to the same number. We will always use a number divisible by all denominators.

Solve Linear Equations Containing Fractions

Example 2 continued:

So, let's use 28 again and change all denominators to 28 as follows:

$$\frac{-2(7)}{4(7)} = \frac{5(2)}{14(2)} - \frac{6(4)}{7(4)}$$

Notice that we achieved a 28 in all denominators by multiplying the existing denominators by an appropriate number. Of course, we had to multiply the numerators by this number as well!

Finally, we will carry out the multiplications to get $\frac{-14}{28} = \frac{10}{28} - \frac{24}{28}$

and $\frac{-14}{28} = \frac{-14}{28}$ or $\frac{-1}{2} = \frac{-1}{2}$ in reduced form.

This is a true statement, therefore the proposed solution $x = 1$ is an actual solution.

5. Cross-Multiplication (1 of 2)

A special method called **Cross-Multiplication** can often be used when we encounter equations in which one fraction is equals to another fraction.

For example, $\frac{F}{3} = \frac{9}{5}$. Incidentally, these type of equations are also called *proportions*!

Cross-Multiplication is a process of multiplying the numerator of one fraction with the denominator of the other fraction on the other side of an equal sign.

Note, we often make a “cross” × over the equal sign to help us remember the *Cross-Multiplication* process!

For example, $\frac{F}{3} = \frac{9}{5}$ and according to the *Cross-Multiplication* process, we can now state ***5F*** = ***3(9)***.

Please note that we wrote ***5F*** and not ***F5*** or ***F(5)***! We always place the coefficient in front of the variable!

Cross-Multiplication (2 of 2)

Let’s finish solving $\frac{F}{3} = \frac{9}{5}$.

Simplifying ***5F*** = ***3(9)***, we get ***5F*** = ***27***. We now solve for ***F*** by dividing both sides of the equal sign by 5 to get

$$F = \frac{27}{5}$$

If we wish, we could carry out a check to prove to ourselves that this is indeed the actual solution.