Concepts
Inverses of Functions

Based on power point presentations by Pearson Education, Inc.

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Learning Objectives

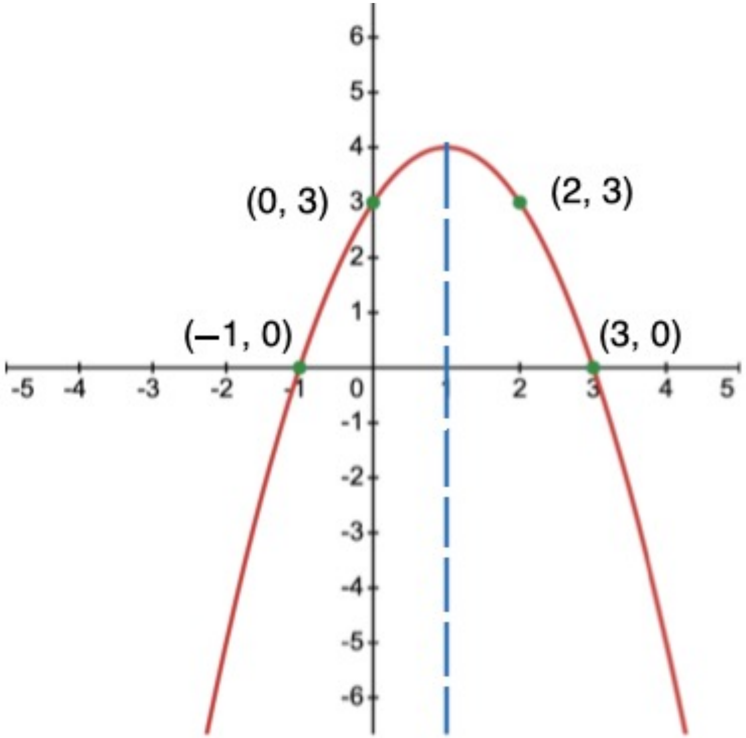
1. Define one-to-one functions.
2. Find inverse functions.
3. Verify inverses.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

1. Definition of One-to-One Functions (1 of 2)

One-to-one function are special function where the range values never repeat.

For example, the quadratic function $\boldsymbol{g(x) = -(x-1)^2 + 4}$ is **NOT** one-to-one. It has repeating range values! Let's look at its graph.

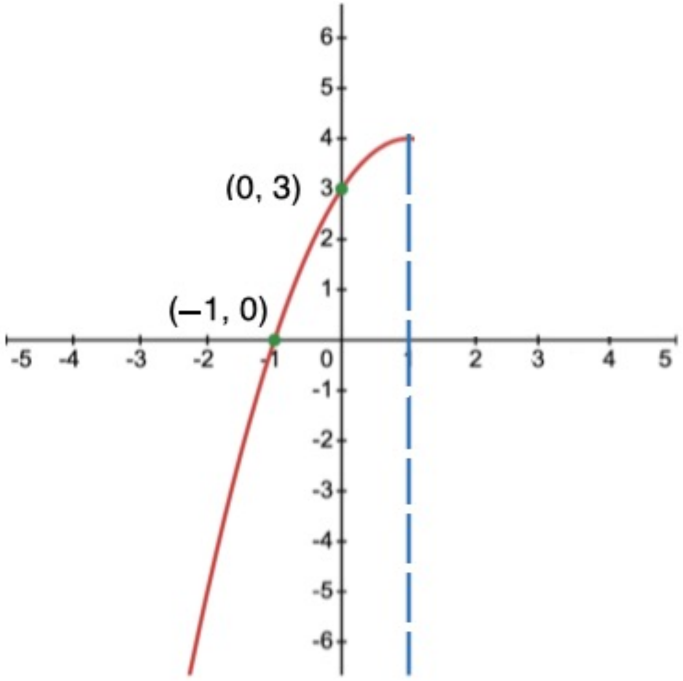


Note that $x = 0$ and $x = 2$ are on opposite side of the *axis of symmetry* and $g(0) = 3$ and $g(2) = 3$. The range value 3 repeats!

You will find that this happens for all *x*-values on opposite side of the axis of symmetry. For example, note that $g(-1) = 0$ and $g(3) = 0$!

Definition of One-to-One Functions (2 of 2)

For example, the domain of $\boldsymbol{g(x) = -(x-1)^2 + 4}$ consists of all real numbers. If we restrict the domain to say $x \leq 1$, then the function becomes one-to-one! Let's look at the graph.

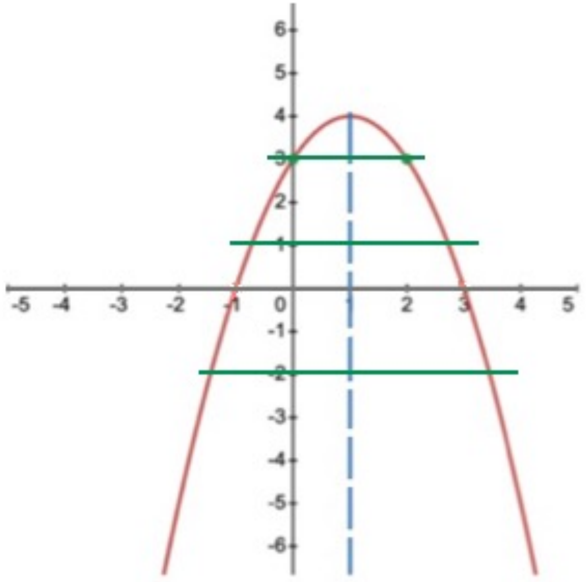


In summary, we can make any function one-to-one by restricting its domain.

Definition of One-to-One Functions (2 of 2)

We can use the *Horizontal Line Test* to quickly determine if a function is one-to-one. Simply graph the function, then draw horizontal lines through the graph. If even just one horizontal line intersects with the graph more than once, the the function is NOT one-to-one.

For example, we can use the *Horizontal Line Test* to show that the function $\boldsymbol{g(x) = -(x-1)^2 + 4}$ is NOT one-to-one since there are many horizontal lines that intersects with the graph more than once.



2. Find the Inverse Function of a Given Function (1 of 3)

Only one-to-one functions have inverse functions! The domain of one is the range of the other and vice versa. The inverses of functions must also be one-to-one.

Strategy for finding inverse functions in *x* and *y*:

Step 1 – Exchange the x and y variables.

Example 1:

The linear function $f(x) = 2x + 7$ is one-to-one. Find its inverse function.

Note that f (x) can be replaced with y. We now interchange x and y to get

$x = 2y + 7$

Find the Inverse Function of a Given Function (2 of 3)

Step 2 – Solve for y in terms of x

Example 1 continued:

$x = 2y + 7$

$x - 7 = 2y$

and $y = \dfrac{x-7}{2}$

Step 3 – Replace y with function notation. Since inverse functions must also be one-to one, their domain might have to be restricted!

Example 1 continued:

Here we use will use ***g*** for the inverse function to get $g(x) = \dfrac{x-7}{2}$ which is a linear function that is always one-to-one.

Find the Inverse Function of a Given Function (3 of 3)

Optional Inverse Function Notation

Sometimes the following notation is used to refer to the inverse function, say of a function ***f***.

$\boldsymbol{f}^{-1}$ which is read as ***"f inverse"***

NOTE: $\boldsymbol{f}^{-1}$ does not mean $\dfrac{\boldsymbol{1}}{\boldsymbol{f}}$.

Example 1 continued:

Instead of ***g*** we can also use $\boldsymbol{f}^{-1}$ to express the inverse function of $f(x) = 2x + 7$ in function notation.

That is, $\boldsymbol{f}^{-1}(x) = \dfrac{x-7}{2}$

3. Verify Inverse Functions (1 of 3)

A function ***f*** and its inverse ***g*** satisfy the following:

$$(f \circ g)(x) = x$$

and

$$(g \circ f)(x) = x$$

Remember that $(f \circ g)$ and $(g \circ f)$ indicate function composition. When ***g*** is an inverse of ***f*** or vice versa, they both equal *x*.

Verify Inverse Functions (2 of 3)

Example 2:

Show that the following functions are inverses:

$$f(x) = 4x - 7 \text{ and } g(x) = \frac{x+7}{4}$$

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f\left(\frac{x+7}{4}\right) \\ &= 4\left(\frac{x+7}{4}\right) - 7 \\ &= x + 7 - 7 \\ &= x\end{aligned}$$

Verify Inverse Functions (3 of 3)

Example 2 continued:

$$(g \circ f)(x) = g(f(x)) = g(4x - 7)$$

$$= \frac{4x - 7 + 7}{4}$$

$$= \frac{4x}{4}$$

$$= x$$

Since both $(f \circ g)$ and $(g \circ f)$ equal *x*, we just verified that the functions ***f*** and ***g*** are inverses.