Concepts
Geometric Sequences and Series

Based on power point presentations by Pearson Education, Inc.

Revised by Ingrid Stewart, Ph.D.

Learning Objectives

1. Memorize the definition of a geometric sequence and find its common ratio.
2. Find the value of a term of a geometric sequence.
3. Memorize the definition of a finite geometric series and evaluate its sum.
4. Evaluate the sum of an infinite geometric series.

NOTE: This lesson contains some examples. You can find more examples in the "Examples" document also located in the appropriate MOM Learning Materials folder.

1. Definition of a Geometric Sequence (1 of 2)

A geometric sequence is another special sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. A geometric sequence can be finite or infinite.

Example 1:

3, 9, 27, 81, 243, … is a geometric sequence because we multiply each preceding term by 3 to get the next term.

Example 2:

142, 146, 150, 154, 158, … is NOT a geometric sequence because there is NO fixed number by which we multiply the preceding term to get the next term. Actually, this is an arithmetic sequence!

Definition of a Geometric Sequence (2 of 2)

The fixed number by which we multiply each preceding term to get the next term in a geometric sequence is called the **common ratio** and is usually denoted by ***r***. It can be positive or negative.

We can find the common ratio by dividing the second term of a geometric sequence by the first term, that is $r = a_2 \div a_1$.

For example, in the geometric sequence 3, 9, 27, 81, 243, … , we would say $r = 9 \div 3 = 3$.

2. The Value of a Term of a Geometric Sequence (1 of 2)

We can find the value of any term a_n of a geometric sequence as long as we know the first term a_1 and common ratio r.

$a_n = a_1 r^{n-1}$ where

a_n is the value of the term to be found
a_1 is the first term of the sequence
n is the position of the unknown term in the sequence (e.g., third term, ninth term)
r is the common ratio

You can find a proof of this formula and those of some other formulas, which we will discuss shortly, in the appropriate weekly tasks folder.

The Value of a Term of a Geometric Sequence (2 of 2)

Example 3:

Find the value of a_7 in a geometric given that the first term a_1 equals 5 and the common difference r is − 3.

Please note that a_7 is the 7th term in the geometric sequence. We will use $a_n = a_1 r^{n-1}$ with $n = 7$.

$a_7 = 5(-3)^{7-1}$

$= 5(-3)^6$

$= 5(729)$ (used the calculator!)

$= 3645$

The value of the 7th term is 3645.

3. The Finite Geometric Series and its Sum (1 of 4)

In general, the **geometric series** is the sum of the terms of a *geometric sequence.* Given a finite geometric series, we can find its sum by writing out the terms and then adding them.

Example 4:

Evaluate the sum of a geometric series $\sum_{k=1}^{5} 6\left(2^{k}\right)$.

$k = 1$ $\quad k = 2$ $\quad k = 3$ $\quad k = 4$ $\quad k = 5$

$$\sum_{k=1}^{5} 6\left(2^{k}\right) = 6\left(2^{1}\right) + 6\left(2^{2}\right) + 6\left(2^{3}\right) + 6\left(2^{4}\right) + 6\left(2^{5}\right)$$

$$= 6(2) + 6(4) + 6(8) + 6(16) + 6(32)$$

$$= 12 + 24 + 48 + 96 + 192$$

$$= 372$$

The Finite Geometric Series and its Sum (2 of 4)

However, we can also use a **Summation Formula** instead of writing out the terms of a finite geometric series and then adding them. This formula is necessary when we are asked to add a large number of terms.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$ where

S_n is the name of the sum

n is the number of terms in the sum

a_1 is the first term of the sum

r is the common ratio

NOTE: The Summation Formula requires us to start with the first term.

The Finite Geometric Series and its Sum (3 of 4)

Example 5:

Evaluate the sum of a geometric series $\sum_{k=1}^{5} 6\left(2^k\right)$ using the Summation Formula $S_n = \frac{a_1\left(1-r^n\right)}{1-r}$.

Let's find the values we need for this formula.

- The number of terms n in the sum is 5.
- We find the first term a_1 by evaluating the series for k = 1.
  Specifically, $a_1 = 6(2^1) = 12$
- We also need r. We find the first and second terms of the geometric series, 12 and 24, then we divide 24 by 12 to find $r = 2$.

The Finite Geometric Series and its Sum

Example 5 continued:

Finally, given a_1 = 12 and r = 2, we can find S_5. Specifically,

$$S_5 = \frac{12\left(1-2^5\right)}{1-2}$$

$$= \frac{12(1-32)}{-1}$$

$$= \frac{12(-31)}{-1}$$

$$= 372$$

4. The Infinite Geometric Series and its Sum (1 of 2)

So far, we have only found the sums of series with a finite upper limit. Usually, we call this finding a **partial sum**.

Sometimes, we can find the sum of an infinite geometric series, that is, a series with an upper limit of infinity. However, this can only be done IF the common ratio *r* is between – 1 and 1.

We can evaluate the sum of an infinite geometric series using the following formula:

$S = \frac{a_1}{1-r}$ where a_1 is the first term of the sum and *r* is the common ratio.

NOTE: For reasons that would be explained in a calculus course, we cannot find sums of any infinite arithmetic series.

The Infinite Geometric Series and its Sum (2 of 2)

Example 6:

Evaluate the sum of an infinite geometric series with a_1 = 5 and $r = \frac{2}{5}$.

We will use the Summation Formula $S = \frac{a_1}{1-r}$. Specifically, we get

$$S = \frac{5}{1-\frac{2}{5}}$$

$$= \frac{5}{\frac{3}{5}}$$

$$= 5 \cdot \frac{5}{3}$$

$$= \frac{25}{3}$$