Operations on Rational Numbers

Based on power point presentations by Pearson Education, Inc.
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Learning Objectives

1. Multiply and divide rational numbers.
2. Add and subtract rational numbers with like denominators.
3. Add and subtract rational numbers with unlike denominators.
4. Perform operations on mixed numbers.
5. Change complex numbers to simple form.

1. Multiply and Divide Rational Numbers

Two rational numbers of the form $\frac{a}{b}$ are **multiplied** by finding the product of their numerators and denominators.

NOTE: The outcome of a multiplication is called **product**!

Two rational numbers of the form $\frac{a}{b}$ are **divided** by finding the product of the first number (called *dividend*) and the **reciprocal*** of the second number (called *divisor*).

NOTE: The outcome of division is called **quotient**!

***Reciprocal** – A fraction in which the numerator and denominator of a given fraction is exchanged.

For example, the reciprocal of $\frac{9}{11}$ is $\frac{11}{9}$. Likewise, the reciprocal of $8 = \frac{8}{1}$ is $\frac{1}{8}$.

Example 1: Multiply Rational Numbers

Find the product of $\left(-\frac{2}{3}\right)\left(-\frac{9}{4}\right)$. If possible, reduce to lowest terms.

Multiply the numbers in the numerator and denominator.

Reduce to lowest terms.

$$\left(-\frac{2}{3}\right)\left(-\frac{9}{4}\right)=\frac{(-2)(-9)}{3\cdot 4}=\frac{18}{12}=\frac{3\cdot 6}{2\cdot 6}=\frac{3}{2}$$

Remember that the product of two negative numbers is positive!

Example 2: Divide Rational Numbers

Find the quotient of $-\frac{3}{5} \div \frac{7}{11}$. If possible, reduce to lowest terms.

Change to multiplication by using the reciprocal of the divisor.

Multiply across.

$$-\frac{3}{5} \div \frac{7}{11} = -\frac{3}{5} \cdot \frac{11}{7} = -\frac{3 \cdot 11}{5 \cdot 7} = -\frac{33}{35}$$

The quotient is reduced to lowest terms.

Remember that the product of a negative and a positive number is negative!

Example 3: Divide Rational Numbers

a. Evaluate $\frac{-9}{0}$

The denominator of any fraction CANNOT contain the value 0. If the denominator of a fraction is 0, we say that the value of this fraction is undefined.

b. Evaluate $\frac{0}{-9}$

A numerator IS allowed to take on the value of 0. Any fraction with a numerator equal to 0 has an overall value of 0, as long as the denominator is not also 0.

c. Evaluate $\frac{0}{0}$

On one hand, its value is undefined because the denominator is equal to ***0***. On the other hand, its value is ***0*** because the numerator is ***0***. Since there is a disagreement, we will say that this value of is **indeterminate**.

NOTE: $\frac{0}{0} \neq 0$

2. Add and Subtract Rational Numbers with Like Denominators

The sum or difference of two or more rational numbers with **like or common denominators** is the sum or difference of their numerators over the like denominator.

Example 4: Add and Subtract Rational Numbers with Like Denominators

Reduce to lowest terms.

a. Find the sum of $\frac{3}{7}+\frac{2}{7}$. If necessary, reduce to lowest terms.

$$\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}$$ This is reduced to lowest terms!

b. Find the difference of $\frac{11}{12}-\frac{5}{12}$. If necessary, reduce to lowest terms.

$$\frac{11}{12}-\frac{5}{12}=\frac{11-5}{12}=\frac{6}{12}$$ This is NOT reduced to lowest terms!

We notice that 6 and 12 have the factor of 6 in common, and when we divide both by this common factor, we get

$$\frac{1}{2}$$

3. Add and Subtract Rational Numbers with Unlike Denominators (1 of 3)

The sum or difference of two or more rational numbers with **unlike denominators** is more cumbersome to calculate. We must first change all denominators to preferably the smallest number they divide into evenly. This is often called the **least common multiple**. Only then can we proceed in finding the sum or difference of the

To find the least common multiple between two numbers there are two methods available. The Guess Method and the Prime Factorization Method.

Add and Subtract Rational Numbers with Unlike Denominators (2 of 3)

1. Guess Method: If the denominators are small, it is often easy to see what the least common multiple (new denominator) should be.

   Example:

   Assume that 4 and 6 are the denominators of two fractions. It is fairly easy to see that their least common multiple, which is the new denominator, must be 12.

2. Prime Factorization Method: Find the prime factors of all denominators and then find the product of all of them. That is your new denominator (least common multiple). HOWEVER, if a prime factor shows up in two or more denominators, and in one denominator it shows up more than once, you must use all occurrences of this prime factor for the new denominator.

Add and Subtract Rational Numbers with Unlike Denominators

Example:

Assume that 56 and 6 are the denominators of two fractions. Let's write them as a product of prime factors:

56 = 2(2)(2)(7)

6 = 2(3)

The prime factors that make up the new denominators consist of 2, 3, and 7. HOWEVER, 2 appears three times in 56 but only once in 6. To take care of the 56, we must use 2 three times in the new denominator.

That is, (2)(2)(2)(3)(7) = 168. That's the new denominator you must use to add these fractions.

Example 5: Add Rational Numbers with Unlike Denominators
(1 of 2)

Find the sum of $\frac{3}{4}+\frac{1}{6}$. If necessary, reduce to lowest terms.

the two rational numbers have unlike denominators, we will first find preferably the smallest number 4 and 6 divide into evenly.

Let's use the Guess Method. We should quickly admit that the smallest number 4 and 6 divide into evenly is 12.

We will now change the denominators 4 and 6 to 12.

$$\frac{3}{4}+\frac{1}{6}=\frac{3}{4}\cdot\frac{3}{3}+\frac{1}{6}\cdot\frac{2}{2}$$

Notice, we must change the numerators as well as the denominators. In effect, we multiplied each rational number by a special version of the number 1.

Example 5: Add Rational Numbers with Unlike Denominators

After multiplication, we end up with $\frac{9}{12} + \frac{2}{12}$.

All that's left to do now is add the numerators and put this sum over the denominator to get

$\frac{11}{12}$ which is reduced to lowest terms.

Example 6: Add Rational Numbers with Unlike Denominators
(1 of 2)

Find the sum of $\frac{3}{4}+\frac{1}{6}$. If necessary, reduce to lowest terms.

Since the two rational numbers have unlike denominators, we will first find preferably the smallest number 4 and 6 divide into evenly.

This time let's use the Prime Factorization Method. We first write 4 and 6 as a product of prime factors:

4 = 2(2)

6 = 2(3)

The prime factors that make up the new denominators consist of 2 and 3. HOWEVER, 2 appears two times in 4 but only once in 6. To take care of the 4, we must use 2 two times in the new denominator.

That is, (2)(2)(3) = 12. That's the new denominator we must use to add these fractions.

Example 6: Add Rational Numbers with Unlike Denominators
(2 of 2)

We will now change the denominators 4 and 6 to 12.

$$\frac{3}{4}+\frac{1}{6}=\frac{3}{4}\cdot\frac{3}{3}+\frac{1}{6}\cdot\frac{2}{2}$$

Notice, we must change the numerators as well as the denominators. In effect, we multiplied each rational number by a special version of the number 1.

After multiplication, we end up with $\frac{9}{12}+\frac{2}{12}$.

All that's left to do now is add the numerators and put this sum over the denominator to get

$\frac{11}{12}$ which is reduced to lowest terms.

Example 7: Add Rational Numbers with Unlike Denominators
(1 of 2)

Find the sum of $\frac{2}{5}+\frac{3}{7}$. If necessary, reduce to lowest terms.

Since the two rational numbers have unlike denominators, we will first find preferably the smallest number 5 and 7 divide into evenly.

We should quickly admit that 5 and 7 are prime numbers and the smallest number they divide into evenly is their product, namely 5(7) = 35.

We will now change the denominators 5 and 7 to 35.

$$\frac{2}{5}+\frac{3}{7}=\frac{2}{5}\cdot\frac{7}{7}+\frac{3}{7}\cdot\frac{5}{5}$$

Notice, we must change the numerators as well as the denominators. In effect, we multiplied each rational number by a special version of the number 1.

Example 7: Add Rational Numbers with Unlike Denominators

(2 of 2)

After multiplication, we end up with $\frac{14}{35} + \frac{15}{35}$.

All that's left to do now is add the numerators and put this sum over the denominator to get

$\frac{29}{35}$ which is reduced to lowest terms.

4. Operations on Mixed Numbers

In order to add, subtract, multiply, and divide mixed numbers, you must first change them to improper fractions. Then you can proceed as per the rules above.

Example 8: Subtracting Mixed Numbers

Find the difference of $5\frac{1}{4} - 2\frac{3}{4}$. Leave the answer as an improper fraction reduced to lowest terms.

Here we must first change the two mixed numbers to improper fractions as discussed earlier.

$$5\frac{1}{4} - 2\frac{3}{4} = \frac{21}{4} - \frac{11}{4}$$

$$= \frac{10}{4}$$ This is NOT reduced to lowest terms!

We notice that 10 and 4 have the factor of 2 in common, and when we divide both by this common factor, we get

$$\frac{5}{2}$$

Example 9: Multiplying Mixed Numbers

Find the product of $5\frac{1}{4}\left(2\frac{3}{4}\right)$ and express as a mixed number.

Here we must first change the two mixed numbers to improper fractions as discussed earlier.

$$5\frac{1}{4}\left(2\frac{3}{4}\right) = \frac{21}{4}\left(\frac{11}{4}\right) = \frac{231}{16}$$

Using the calculator, we find that 16 divides into 231 fourteen times with a remainder of 7.

Therefore, $\frac{231}{16} = 14\frac{7}{16}$.

5. Complex Numbers

Numbers with numerators and/or denominators that are not integers are called complex numbers. In mathematics, we are required to change these complex numbers so that only integers are in their numerators and denominators.

Some examples:

$$\frac{1}{\frac{1}{14}} \quad \text{or} \quad \frac{\frac{2}{7}}{\frac{1}{14}}$$

Example 10: Convert Complex Numbers

Convert $\frac{1}{\frac{1}{14}}$ to a number only containing an integer in the numerator and the denominator.

We see that there is the rational number 1 in the numerator and the rational number $\frac{1}{14}$ in the denominator.

We know that horizontal lines separating two numbers usually indicate division. We also know that if rational numbers are divided, we simply find the product of the first number and the reciprocal of the second number.

Therefore, we will write the following: $\frac{1}{\frac{1}{14}} = 1 \div \frac{1}{14} = 1 \cdot \frac{14}{1} = \frac{14}{1}$

We now have the integers 14 and 1 in the numerator and denominator, respectively. Of course, we can simply express the product as 14.

Example 11: Convert Complex Rational Numbers

Convert $\frac{\frac{2}{7}}{\frac{1}{14}}$ to a rational number containing only an integer in the numerator and the denominator.

We see that there is the rational number $\frac{2}{7}$ in the numerator and the rational number $\frac{1}{14}$ in the denominator.

We know that horizontal lines separating two numbers usually indicate division. We also know that if rational numbers are divided, we simply find the product of the first number and the reciprocal of the second number.

Therefore, we will write the following: $\frac{\frac{2}{7}}{\frac{1}{14}} = \frac{2}{7} \div \frac{1}{14} = \frac{2}{7} \cdot \frac{14}{1} = \frac{28}{7}$

We now have the integers 28 and 7 in the numerator and denominator, respectively. Of course, we can simply express the product as 4.