Introduction to Rational Numbers

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Learning Objectives

1. Define the rational numbers.
2. Define fractions.
3. Reduce fractions to lowest terms.
4. Convert mixed numbers to improper fractions and vice versa.
5. Express terminating/non-repeating decimal numbers as fractions.
6. Express non-terminating/repeating decimal numbers as fractions.
7. Express fractions as decimal numbers.

1. Definition of Rational Numbers 1 of 3

We know that the set of *Natural Numbers* consists of the following numbers:

{1, 2, 3, 4, 5, 6, …}

The set of *Integers* is defined as the *natural numbers*, plus the number *0*, plus the *negatives of the natural numbers*.

That is, {…, – 3, – 2, – 1, 0, 1, 2, 3, …}.

The set of ***Rational Numbers*** consists of all numbers which can be expressed in the form $\frac{a}{b}$, where *a* and *b* are *integers* and *b* is not equal to *0*.

Definition of Rational Numbers (2 of 3)

The horizontal line between *a* and *b* indicates division. That is, $\frac{a}{b}$ can be written as $a \div b$.

The integer *a* in $\frac{a}{b}$ is called the ***numerator***.

The integer *b* is called the ***denominator***.

Sometimes a slant line / replaces the horizontal line, and the rational number is written as *a/b*. However, this is a less common way of expressing a rational number.

Definition of Rational Numbers (3 of 3)

Please note that integers are considered rational numbers because they can be written as $\frac{a}{b}$.

For example, the number 8 can be written as a rational number in many different ways. For example, it can be expressed as $\frac{8}{1}, \frac{16}{2}, \frac{32}{4}$ etc.

Likewise, the number 1 can be expressed as a rational number in many different ways. For example, it can be written as $\frac{1}{1}, \frac{2}{2}, \frac{100}{100}$, etc.

Also, many decimal numbers can be expressed as rational numbers. We will show later that there is a way to express certain decimal numbers in the form $\frac{a}{b}$.

2. Definition of a Fraction (1 of 2)

While we usually call every number $\frac{a}{b}$ a fraction, there is actually a difference between fractions and rational numbers.

Only those rational numbers in which ***a*** and ***b*** are *natural numbers* should be called fractions.

For example, $\frac{-3}{4}$ is a rational number but not technically a fraction because its numerator is not a natural number.

Definition of a Fraction (2 of 2)

Proper Fractions

Fractions in which the numerator is less than the denominator are called *proper fractions*.
For example, $\frac{3}{4}$.

Improper Fractions

Fractions in which the numerator is greater than the denominator are called *improper fractions*.
For, example, $\frac{8}{3}$.

Mixed Numbers

Numbers that have a whole part and a fractional part that is a proper fraction are called *mixed numbers*.

For example, $2\frac{2}{3}$.

3. Reduce Fractions to Lowest Terms

To reduce fractions to lowest terms, we use either the **Prime Factorization Method** or the **"Educated Guessing" Method**. For both methods, it is important to know the *tests for divisibility*.

Then we "cancel". That is, we cross out each prime number in the numerator with a match in the denominator. If there is no match, the prime numbers cannot be "canceled" and must stay in the numerator and/or denominator.

Example 1: Reduce Fractions to Lowest Terms

Reduce $\frac{12}{80}$ to lowest terms using the *Prime Factorization Method*.

When we write the numerator and denominator as products of their prime factors, we get the following:

$$\frac{12}{80} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$$

We now cross out (cancel) each prime number in the numerator with a match in the denominator as follows:

$$\frac{12}{80} = \frac{\cancel{2} \cdot \cancel{2} \cdot 3}{\cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 5}$$

Please note that the remaining prime numbers have no more matches. Therefore, the fraction is reduced to lowest terms.

That is, $\frac{12}{80} = \frac{3}{2 \cdot 2 \cdot 5} = \frac{3}{20}$

Example 2: Reduce Fractions to Lowest Terms

Let's reduce $\frac{12}{80}$ again, but this time we will use the *Educated Guessing Method*.

Using the *tests for divisibility* as a guide, we just happen to notice that ***4*** divides evenly into the numerator and the denominator.

$$\frac{\overset{3}{\cancel{12}}}{\underset{20}{\cancel{80}}} = \frac{3}{20}$$

Voilà, we already ended up with the rational number reduced to lowest terms as shown in Example 1.

4. Convert Mixed Numbers to Improper Fractions and Vice Versa

Mixed numbers and improper fractions are related. We can convert improper fractions to mixed numbers as follows:

1. Divide the denominator into the numerator to find the integer quotient and remainder.
2. Write the integer remainder over the denominator and place next to the integer quotient.

We can convert mixed numbers to improper fractions as follows:

1. Multiply the denominator of the fractional part of the mixed number by the integer part then add the numerator of the fractional part to this product.
2. Place the sum in step 1 over the denominator of the mixed number.

Example 3: Convert a Mixed Number to an Improper Fraction

Change $3\frac{1}{2}$ to an improper fraction.

NOTE: $3\frac{1}{2}$ means $3+\frac{1}{2}$!!!!!!!!!!!

We multiply the whole number ***3*** by the denominator of the fractional part, which is ***2***.

To this product we then add the numerator of the fractional part which is ***1***. The denominator stays the same throughout.

$$3\frac{1}{2} = \frac{3 \cdot 2 + 1}{2} = \frac{6+1}{2} = \frac{7}{2}$$

Example 4: Convert an Improper Fraction to a Mixed Number

Write the improper fraction $\frac{8}{3}$ as a mixed number with a fractional part.

We should know that 3 divides into 8 twice with a remainder of 2.

Quickly we should be able to state $\frac{8}{3} = 2\frac{2}{3}$.

Example 5: Convert an Improper Fraction to a Mixed Number

Write the improper fraction $\frac{235}{17}$ as a mixed number with a fractional part.

Here we use the calculator to help with the task. We find that 17 divides into 235 thirteen-times (13) with some decimal part. The integer quotient is 13.

We can convert the decimal part to an integer part as follows:

$$235 - 17(13) = 14$$

You will now use the integer quotient ***13*** and remainder ***14*** as follows:

$$\frac{235}{17} = 13\frac{14}{17}$$

5. Express Terminating/Non-Repeating Decimal Numbers as Fractions

Earlier we learned that decimal numbers can be terminating/non-repeating, non-terminating/repeating, or non-terminating/non-repeating.

Following is a strategy for changing terminating/non-repeating decimal numbers into fractions:

1. Write down the decimal divided by 1.
2. Multiply both top and bottom by an integer multiple of 10. The integer depends on the number of decimal places!
3. Reduce the fraction, if necessary.

Example 6: Express Terminating/Non-Repeating Decimals as Fractions

Express ***0.7*** as a fraction. If necessary, reduce to lowest terms

Write $\frac{0.7}{1}$

There is one (1) number after the decimal point, therefore, multiply both the numerator and the denominator by 1(10). Use a calculator if necessary!

$\frac{0.7}{1} \cdot \frac{10}{10} = \frac{7}{10}$ This is reduced to lowest terms!

Example 7: Express Terminating/Non-Repeating Decimals as Fractions

Express ***5.49*** as a fraction. If necessary, reduce to lowest terms.

Write $\frac{5.49}{1}$

There are two (2) numbers after the decimal point, therefore, multiply both the numerator and the denominator by 2(10) = 100. Use a calculator if necessary!

$\frac{5.49}{1} \cdot \frac{100}{100} = \frac{549}{100}$ This is reduced to lowest terms!

Example 8: Express Terminating/Non-Repeating Decimals as Fractions (1 of 2)

Express ***0.125*** as a fraction. If necessary, reduce to lowest terms.

Write $\frac{0.125}{1}$

There are three (3) numbers after the decimal point, therefore, multiply both the numerator and the denominator by 3(10) = 1000. Use a calculator if necessary!

$$\frac{0.125}{1} \cdot \frac{1000}{1000} = \frac{125}{1000}$$

This is NOT reduced to lowest terms!

Using the *tests for divisibility* as a guide, we notice that the numerator ends in 5 and the denominator in 0. This means that ***5*** divides into both.

$$\frac{\overset{25}{\cancel{125}}}{\underset{200}{\cancel{1000}}} = \frac{25}{200}$$

Example 8: Express Terminating/Non-Repeating Decimals as Fractions (2 of 2)

However, we are not done. We notice that ***5*** still divides into both the numerator and the denominator.

$$\frac{\overset{5}{\cancel{25}}}{\underset{40}{\cancel{200}}} = \frac{5}{40}$$

However, we are still not done. We again notice that ***5*** divides into both the numerator and the denominator.

$$\frac{\overset{1}{\cancel{5}}}{\underset{8}{\cancel{40}}} = \frac{1}{8}$$

This is now reduced to lowest terms! The numerator and the denominator no longer have factors in common.

We found that 0.125 is equal to $\frac{1}{8}$.

6. Express Non-Terminating/Repeating Decimal Numbers as Fractions

Examples of non-terminating, repeating decimals are

0.6666666666 … or ***0.818181818181818181 …*** . Both can be written as follows:

$0.\overline{6}$ and $0.\overline{81}$ with horizontal bars over the number or numbers that repeat indefinitely.

Normally, we would have to go through a lengthy (and boring) process to find its equivalent fraction. We are not going to do this here. Instead, we will be discussing the following shortcut method.

1. The denominator of the fraction will have as many 9's as the number of repeating digits.
2. The numerator will be the repeating number without the decimal or the bar.
3. Reduce the fraction to lowest terms, if necessary.

Example 9: Express Non-Terminating/Repeating Decimals as Fractions

Express $0.\overline{6}$ as a fraction reduced to lowest terms.

There is one (1) repeating digit. Therefore, the denominator of the fraction becomes 9. The numerator will be the repeating number without the decimal and the bar.

$0.\overline{6} = \frac{6}{9}$ This is NOT reduced to lowest terms!

Using the *tests for divisibility* as a guide, we notice that the numerator and denominator are divisible by ***3***.

Finally, we can state $0.\overline{6} = \frac{2}{3}$ which is reduced to lowest terms.

Example 10: Express Non-Terminating/Repeating Decimals as Fractions (1 of 2)

Express $2.\overline{81}$ as an improper fraction reduced to lowest terms.

Let's work on the decimal portion first and then worry about the whole number.

There are two (2) repeating digits in the decimal portion. Therefore, the denominator of *a/b* becomes 99. The numerator will be the repeating number without the decimal and the bar.

$0.\overline{81} = \frac{81}{99}$ This is NOT reduced to lowest terms!

Using the *tests for divisibility* as a guide, we notice that the numerator and denominator are divisible by ***9***. Therefore, ***9*** divides both into the numerator and the denominator.

Finally, we can state $0.\overline{81} = \frac{9}{11}$ which is reduced to lowest terms.

Example 10: Express Non-Terminating/Repeating Decimals as Fractions (2 of 2)

We now know that $2.\overline{81}$ can be written as $2\frac{9}{11}$ and we can change this to an improper fraction.

We multiply the whole number ***2*** by the denominator of the fractional part which is ***11***.

To this product we then add the numerator of the fractional part, which is ***9***. The denominator stays the same throughout.

$$2\frac{9}{11} = \frac{2 \cdot 11 + 9}{11} = \frac{31}{11}$$

7. Express Fractions as Decimal Numbers

Any fraction can be expressed as a decimal number by dividing the denominator into the numerator.

The easiest way to change a fraction into a decimal number is by using a calculator. You simply divide the numerator by the denominator.

Example 11: Express Fractions as Decimal Numbers

Express the following fractions as decimal numbers using the calculator. If necessary, round to 3 decimal places.

a. $\frac{1}{10}$

Calculator Tip:

1	÷	10	Enter

$\frac{1}{10} = 0.1$ There is no need to round!

b. $\frac{5}{8}$

$\frac{5}{8} = 0.625$ There is no need to round!

Example 12: Express Fractions as Decimal Numbers

Express $\frac{1}{3}$ as a decimal number using the calculator. If necessary, round to 3 decimal places.

Calculator Tip:

1	÷	3	Enter

We notice that the calculator shows us seemingly infinitely many repeating 3's after the decimal point. That is, we encountered a **not terminating and repeating decimal number**.

We are asked to round it to to decimal places to get 0.333. Of course, we could have also written $0.\overline{3}$.