DETAILED SOLUTIONS AND CONCEPTS - SOLVING TRIGONOMETRIC EQUATIONS - PART 1

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Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Strategy for Solving Equations of the Form

$sin(ax) = C$
$cos(ax) = C$
$tan(ax) = C$, **where $a = 1$ and the solutions are restricted**

Step 1:

If necessary, isolate the trigonometric expression on one side of the equation. **This let's you know the sign of the numeric value (positive or negative).** Then use the *Inverse Trigonometric Function* concept to find a solution for the angle x with the calculator.

HINT: It is easier to find this angle in degrees! Even if you are asked for solutions in radians, you might want to find their degree equivalent first, then change the final answer to radians!!!

Step 2:

Find the Reference Angle α for the angle x, if it exists. Remember that Quadrantal Angles do NOT have Reference Angles.

Step 3:

Find the solution(s) for the angle x using the appropriate solution interval and **All Students Take Calculus**.

You must be able to explain the solution process (see examples below).

Problem 1:

Solve $\sin x = \frac{\sqrt{2}}{2}$ for x on the interval $[0,2\pi)$. Express your answers in **EXACT** radians

Please note that all trigonometric equations have infinitely many solutions! However, in this case the solutions are restricted to the interval $[0,2\pi)$. That is, you MUST list only solutions that fall within it.

Step 1:

We are going to use the concept of *Inverse Trigonometric Functions* to solve for the angle x using the calculator.

$$\sin x = \frac{\sqrt{2}}{2}$$

The numeric value is positive !

$$x = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$x = 45^{\circ}$$

NOTE: It's easier to use degrees in the solution process! Then, for your final answer, change the solutions to radians!

Step 2:

Reference Angle for angle $x = 45^{\circ}$ is $\alpha = 45^{\circ}$.

Step 3:

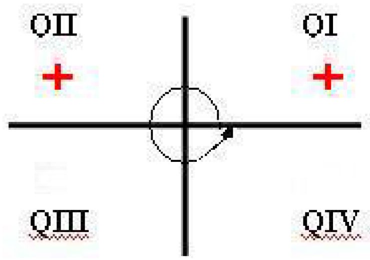
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **positive**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of sine is **positive** for angles in QI and QII (see Step 1). Given a reference angle of 45°, the solutions for x on the interval $[0,2\pi)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 45° \equiv \frac{\pi}{4}$$

$$x_2 = 180° - 45° = 135° \equiv \frac{3\pi}{4}$$

Problem 2:

Solve $\sin x = -\frac{\sqrt{2}}{2}$ for x on the interval $[0,2\pi)$. Express your answers in **EXACT** radians

Step 1:

We are going to use the concept of *Inverse Trigonometric Functions* to solve for the angle x using the calculator.

$\sin x = -\frac{\sqrt{2}}{2}$ **The numeric value is negative !**

$$x = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

$$x = -45°$$

NOTE: It's easier to use degrees in the solution process!
Then, for your final answer, change the solutions to radians!

Step 2:

Reference Angle for angle $x = -45^o$ is $\alpha = 45^o$.

Please note that the range of the arcsine is limited to the interval $[-90°, 90°]$. That is why the calculator gives us ***- 45º*** **and not** ***45º*****.**

Step 3:

Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **negative**.

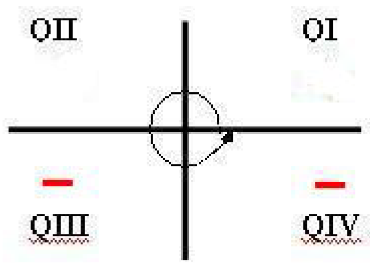
See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of sine is **negative** for angles in QIII and QIV (see Step 1). Given a reference angle of 45^o, the solutions for x on the interval $[0,2\pi)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 225° \equiv \frac{5\pi}{4}$$
$$x_2 = 315° \equiv \frac{7\pi}{4}$$

Problem 3:

Solve $\tan x = \sqrt{3}$ for x on the interval $[0,2\pi)$. Express your answers in **EXACT** radians.

Step 1:

$\tan x = \sqrt{3}$ **The numeric value is positive !**

and using the calculator, we find

$$x = \tan^{-1}\left(\sqrt{3}\right)$$
$$x = 60°$$

NOTE: It's easier to use degrees in the solution process! Then, for your final answer, change the solutions to radians!

Step 2:

Reference Angle for angle $x = 60^o$ is $\alpha = 60^o$.

Step 3:

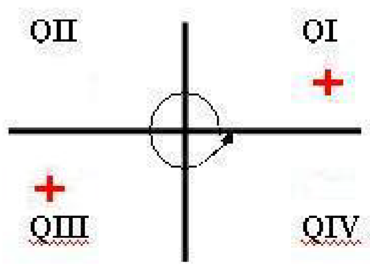
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **positive**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of tangent is **positive** for angles in QI and QIII (see Step 1). Given a reference angle of 60^o, the solutions for x on the interval $[0°,360°)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 60° \equiv \frac{\pi}{3}$$

$$x_2 = 240° \equiv \frac{4\pi}{3}$$

Problem 4:

Solve $\tan x = -\sqrt{3}$ for x on the interval $[0,2\pi)$. Express your answers in **EXACT** radians.

Step 1:

$\tan x = -\sqrt{3}$ **The numeric value is negative !**

and using the calculator, we find

$$x = \tan^{-1}\left(-\sqrt{3}\right)$$
$$x = -60^\circ$$

NOTE: It's easier to use degrees in the solution process! Then, for your final answer, change the solutions to radians!

Step 2:

Reference Angle for angle $x = -60^o$ is $\alpha = 60^o$.

Please note that the range of the arctangent is limited to the interval $(-90^\circ, 90^\circ)$. That is why the calculator gives us *- 60^o* and not *60^o*.

Step 3:

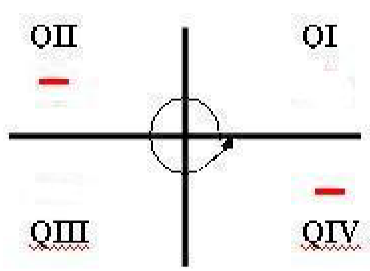
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **negative**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of tangent is **negative** for angles in QII and QIV (see Step 1). Given a reference angle of 60^o, the solutions for x on the interval $[0, 2\pi)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 120^\circ \equiv \frac{2\pi}{3}$$

$$x_2 = 300^\circ \equiv \frac{5\pi}{3}$$

Problem 5:

Let's solve $\tan x = -\sqrt{3}$ for x again, however, this time on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Be sure to express your answer in **EXACT** radians.

This example helps to illustrate how the solutions to trigonometric equations can vary depending on the solution interval! Please compare with Example 4.

Step 1:

$\tan x = -\sqrt{3}$ **The numeric value is negative !**

and using the calculator, we find

$$x = \tan^{-1}\left(-\sqrt{3}\right)$$
$$x = -60^\circ$$

NOTE: It's easier to use degrees in the solution process!
Then, for your final answer, change the solutions to radians!

Step 2:

Reference Angle for angle $x = -60^o$ is $\alpha = 60^o$.

Step 3:

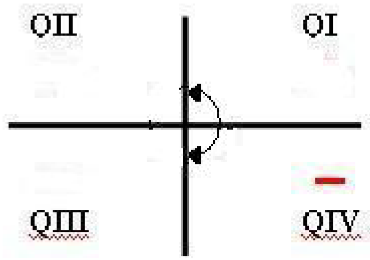
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **negative**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of tangent is **negative** for angles in QII and QIV (see Step 1). Given a reference angle of 60^o, the solutions for x on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, there is only one solution which is

$$x = -60° \equiv -\frac{\pi}{3}$$

Please note that angles larger than $\pi/2$ or smaller than $-\pi/2$ are NOT in the solution set, although some of them might be "perfectly good" solutions!

Problem 6:

Solve $\cos x = \frac{\sqrt{3}}{2}$ for x on the interval $[0°, 360°)$. Express your answers in degrees.

Step 1:

$\cos x = \frac{\sqrt{3}}{2}$ **The numeric value is positive !**

and using the calculator, we find

$$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$x = 30°$$

Step 2:

Reference Angle for angle $x = 30^o$ is $\alpha = 30^o$.

Step 3:

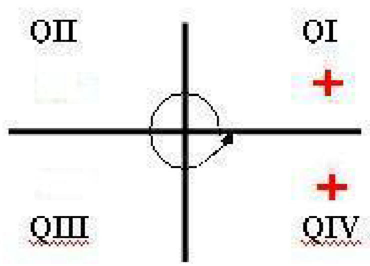
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **positive**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of cosine is **positive** for angles in QI and QIV (see Step 1). Given a reference angle of 30^o, the solutions for x on the interval $[0^\circ, 360^\circ)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 30^\circ$$
$$x_2 = 330^\circ$$

Problem 7:

Solve $\cos x = -0.67$ for x on the interval $[0, 2\pi)$. Express your answers in radians.

Please note that -0.67 is NOT a numeric value associated with our "special" angles. In this case I CANNOT ask you to find EXACT solutions!

In this case, we will find the answers in radians immediately without using degrees first because the rounding error gets too large otherwise.

It make the calculations easier, let's agree to round ALL calculations to 2 decimal places. However, let's use π instead of 3.14.

Step 1:

$\cos x = -0.67$ **The numeric value is negative !**

and using the calculator, we find

x = arccos (-0.67)

$x \approx 2.31$ (this is a QII angle!)

Step 2:

Reference Angle for angle $x \approx 2.31$ is $\alpha \approx \pi - 2.31 = 0.83$.

Step 3:

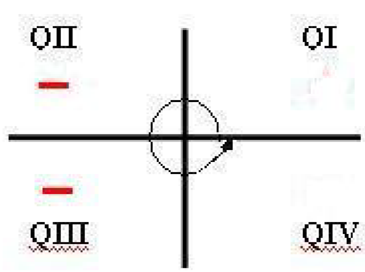
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **negative**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of cosine is **negative** for angles in QII and QIII (see Step 1). Given a reference angle of 0.83, the solutions for x on the interval $[0,2\pi)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$x_1 \approx 2.31$

$x_2 \approx \pi + 0.83 = 3.97$

Problem 8:

Solve $\sin x = 0.15$ for x on the interval $[0,2\pi)$. Express your answers in radians.

Please note that *0.15* is NOT a numeric value associated with our "special" angles. In this case I CANNOT ask you to find EXACT solutions!

In this case, we will find the answers in radians immediately without using degrees first because the rounding error gets too large otherwise.

It make the calculations easier, let's agree to round ALL calculations to 2 decimal places. However, let's use π instead of *3.14*.

Step 1:

$\sin x = 0.15$ **The numeric value is positive !**

and using the calculator, we find

$x = \arcsin(0.15)$

$x \approx 0.15$ (this is a QI angle!)

Please note that it can happen that the numeric value and the angle (in radians) are approximately the same value! Just remember that ***0.15 radians*** is approximately 8.63°.

Step 2:

Reference Angle for angle $x \approx 0.15$ is $\alpha \approx 0.15$.

Step 3:

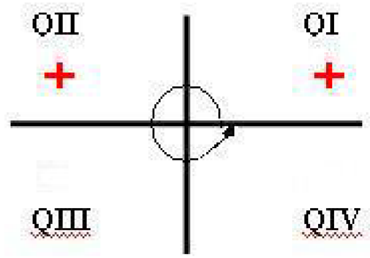
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **positive**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of sine is **positive** for angles in QI and QII (see Step 1). Given a reference angle of 0.15, the solutions for x on the interval $[0,2\pi)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$x_1 \approx 0.15$

$x_2 \approx \pi - 0.15 = 2.99$

Problem 9:

Solve $\tan x = -0.45$ for x on the interval $[0,2\pi)$. Express your answers in radians.

Please note that -0.45 is NOT a numeric value associated with our "special" angles. In this case I CANNOT ask you to find EXACT solutions!

In this case, we will find the answers in radians immediately without using degrees first because the rounding error gets too large otherwise.

It make the calculations easier, let's agree to round ALL calculations to 2 decimal places. However, let's use π instead of 3.14.

Step 1:

$\tan x = -0.45$ **The numeric value is negative !**

and using the calculator, we find

$x = \arctan(-0.45)$

$x \approx -0.42$ (this is a QIV angle!)

Step 2:

Reference Angle for angle $x \approx -0.42$ is $\alpha \approx 0.42$.

Step 3:

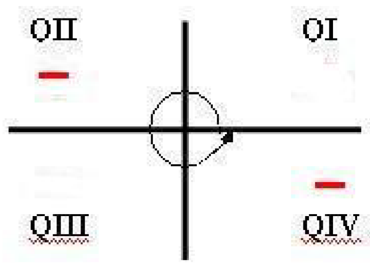
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **negative**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of tangent is **negative** for angles in QII and QIV (see Step 1). Given a reference angle of ***0.42***, the solutions for ***x*** on the interval $[0, 2\pi)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 \approx \pi - 0.42 = 2.72$$

$$x_2 \approx 2\pi - 0.42 = 5.86$$

Problem 10:

Solve $\cos x = 0.37$ for ***x*** on the interval $[0°, 360°)$. Express your answers in degrees.

Please note that *0.37* is NOT a numeric value associated with our "special" angles. In this case I CANNOT ask you to find EXACT solutions!

It make the calculations easier, let's agree to round ALL calculations to 2 decimal places.

Step 1:

$\cos x = 0.37$ **The numeric value is positive !**

and using the calculator, we find

$$x = \arccos(0.37)$$

$x \approx 68.28^o$ (this is a QI angle!)

Step 2:

Reference Angle for angle $x \approx 68.28^o$ is $\alpha \approx 68.28^o$.

Step 3:

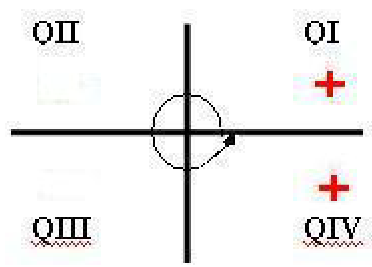
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is **positive**. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of cosine is **positive** for angles in QI and QIV (see Step 1). Given a reference angle of 68.28^o, the solutions for x on the interval $[0°, 360°)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 \approx 68.28^o$$

$$x_2 \approx 360^o - 68.28^o = 291.72^o$$

Problem 11:

Solve $\sin x = 1$ for x on the interval $[0°, 720°)$. Express your answers in degrees.

Step 1:

$$\sin x = 1$$

and using the calculator, we find

$$x = \sin^{-1}(1)$$
$$x = 90°$$

Step 2:

$90°$ is a Quadrantal Angle, which has no Reference Angle!

Step 3:

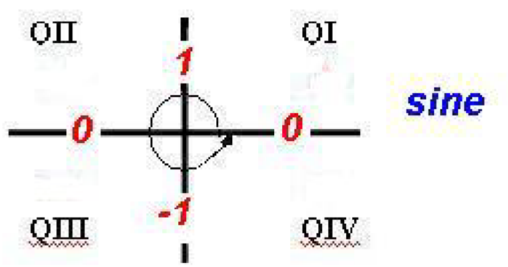
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is 1. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of sine is 1 only for $90°$ in the interval $[0°, 360°)$. Therefore, the solutions for x on the interval $[0°, 720°)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 90°$$
$$x_2 = 360° + 90° = 450°$$

Problem 12:

Solve $\cos x = 0$ for x on the interval $[0°, 360°)$. Express your answers in degrees.

Step 1:

$$\cos x = 0$$

and using the calculator, we find

$$x = \cos^{-1} 0$$
$$x = 90°$$

Step 2:

$90°$ is a Quadrantal Angles, which has no Reference Angle!

Step 3:

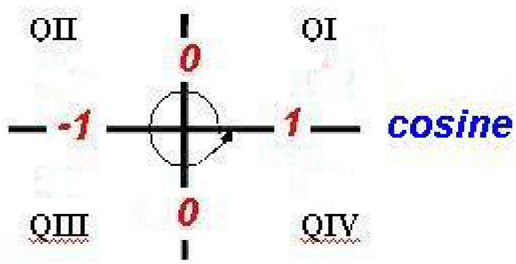
Here we are going to use the fact that once the trigonometric expression is isolated on one side, the numeric value is 0. See Step 1 above.

You must explain the solution process. You can either use a sentence ...

For example:

The numeric value of cosine is 0 for $90°$ and $270°$ in the interval $[0°,360°)$. Therefore, the solutions for x on the interval $[0°,360°)$ are as follows.

or you can show a picture showing the solution interval and the quadrants in which we have to find the angles ...



In any case, the solutions are

$$x_1 = 90°$$
$$x_2 = 270°$$