



DETAILED SOLUTIONS AND CONCEPTS - QUADRATIC FUNCTIONS
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Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Standard Equation

$$f(x) = a(x - h)^2 + k$$

General Equation

$$f(x) = ax^2 + bx + c$$

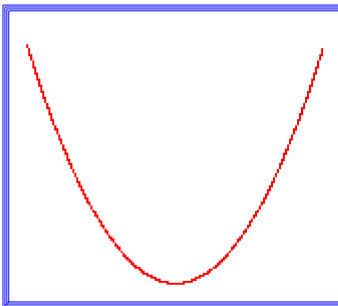
Where $a, b, c, h,$ and k are real numbers and $a \neq 0$.

The domain consists of All Real Numbers.

Characteristics of Graphs of Quadratic Functions

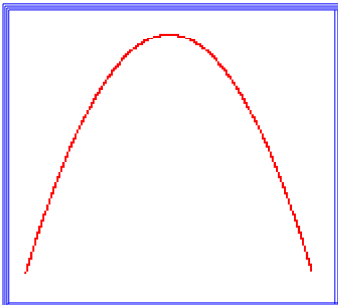
The graphs of quadratic functions are so popular that they were given their own name. They are called **parabolas**.

- The graph is SMOOTH and symmetric to a line called the **axis of symmetry**. This means that the two sides of the parabola on either side of the *axis of symmetry* look like mirror images of each other.
- The point where the *axis of symmetry* intersects with the *parabola*, is called the **vertex**.
- The graph is U-shaped at the vertex.
- The graph is NEVER parallel to the y-axis. Instead it moves away from it at a steady pace.
- There is always a y-intercept.
- There are at most two x-intercepts. This means that some parabolas may have no x-intercept, while others may have one or two x-intercepts.



The graph of a quadratic function $f(x) = ax^2 + bx + c$ opens upward when $a > 0$.

In that case, the *vertex* is the **minimum** point on the graph. That is, there is no other point on the graph whose y-value is lower.



The graph of a quadratic function $f(x) = ax^2 + bx + c$ opens downward when $a < 0$.

In that case, the *vertex* is the **maximum** point on the graph. That is, there is no other point in the graph whose y-value is higher.

Given a quadratic function in the form $f(x) = a(x - h)^2 + k$:

- The equation of the *axis of symmetry* is $x = h$.
- The coordinates of the *vertex* are (h, k) .

Given a quadratic function in the form $f(x) = ax^2 + bx + c$:

- The equation of the *axis of symmetry* is $x = -\frac{b}{2a}$. The b and the a come from $f(x) = ax^2 + bx + c$.

- The coordinates of the *vertex* are $\left[-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right]$.

Strategy for Graphing Quadratic Equations

- Create a *Cartesian Coordinate System*.
- Find and plot the coordinates of the vertex.
- Find and plot the *Axis of Symmetry* as a dashed vertical line.
- Find and plot the the x-intercept(s) (there might be none, one, or two).
- Find and plot the the y-intercept (there is always exactly one).
- Find and plot 3-5 other points to better facilitate the shape of the graph, particularly the U-shaped vertex!
- Connect all points in the shape of a parabola.

Problem 1:

Given $g(x) = (x - 2)^2 + 1$, do the following:

- Find the coordinates of the vertex
- Find the equation of the Axis of Symmetry
- Find the coordinates of the x-intercept(s)
- Find the coordinates of the y-intercept(s)
- Graph the function.

- Coordinates of the Vertex:

This quadratic function is in standard form. We do not have to do any calculations to find the coordinates of the vertex. We get the information straight from the equation.

The coordinates are $(2, 1)$.

- Equation of the *Axis of Symmetry*:

$$x = 2$$

- Coordinates of the x-intercept(s):

$$0 = (x - 2)^2 + 1$$

$$(x - 2)^2 = -1$$

$$x - 2 = \pm\sqrt{-1}$$

$$x = \pm i + 2$$

Since we are encountering imaginary numbers we can conclude that there are NO x-intercepts.

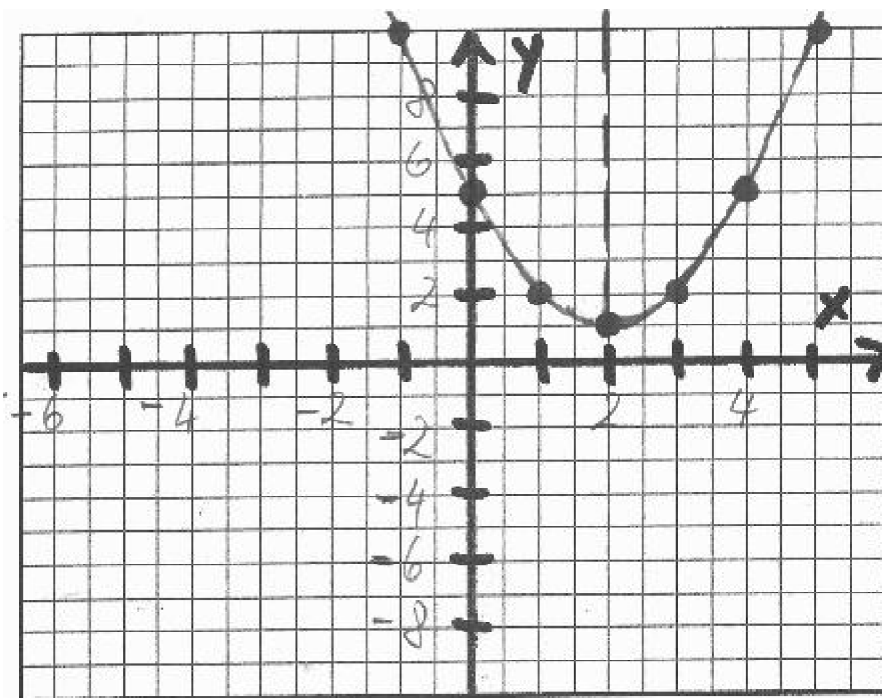
- Coordinates of the y-intercept:

$$g(0) = (0 - 2)^2 + 1 = 5$$

The coordinates of the y-intercept are $(0, 5)$.

- Now, let's look at the graph:

- The graph is SMOOTH and the vertex is U-shaped.**
- The graph is NEVER parallel to the y-axis. Instead it moves away from it at a steady pace.**



Problem 2:

Given $f(x) = -(x + 2)^2 - 1$, do the following:

- Find the coordinates of the vertex
- Find the equation of the Axis of Symmetry
- Find the coordinates of the x-intercept(s)
- Find the coordinates of the y-intercept(s)
- Graph the function.

- Coordinates of the Vertex:

This quadratic function is in standard form. We do not have to do any calculations to find the coordinates of the vertex. We get the information straight from the equation.

In our case, $(x + 2) = [x - (-2)]$, and the x-coordinate of the vertex becomes **-2**.

The vertex is at **$(-2, -1)$** .

- Equation of the *Axis of Symmetry*:

$$x = -2$$

- Coordinates of the x-intercept(s):

$$0 = -(x + 2)^2 - 1$$

$$(x + 2)^2 = -1$$

$$x + 2 = \pm\sqrt{-1}$$

$$x = \pm i - 2$$

Since we are encountering imaginary numbers we can conclude that there are NO x-intercepts.

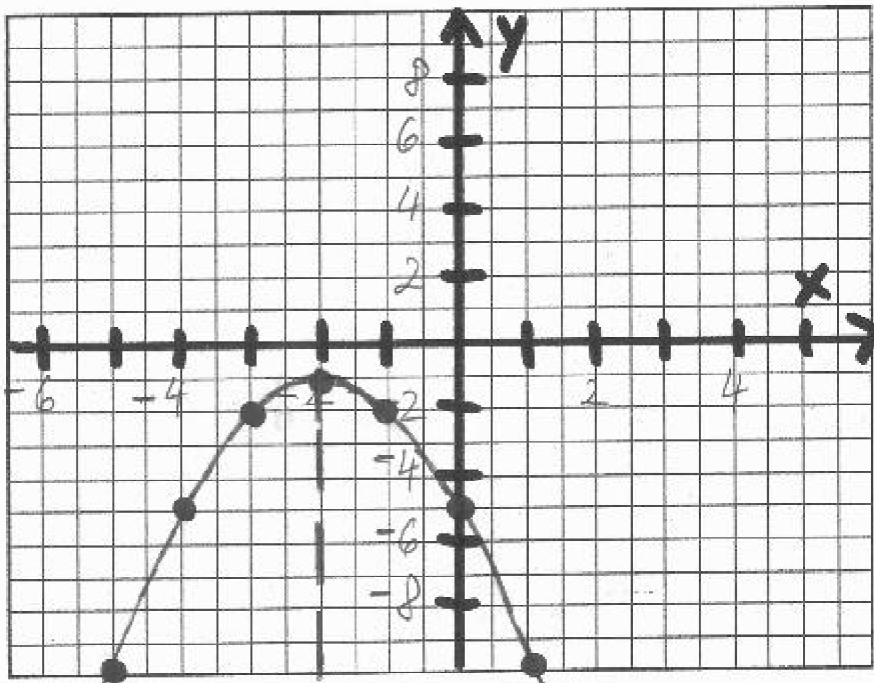
- Coordinates of the y-intercept:

$$f(0) = -(0 + 2)^2 - 1 = -5$$

The coordinates of the y-intercept are $(0, -5)$.

- Now, let's look at the graph:

1. The graph is **SMOOTH** and the vertex is **U-shaped**.
2. The graph is **NEVER** parallel to the y-axis. Instead it moves away from it at a steady pace.



Problem 3:

Given $k(x) = x^2 - 2x + 1$, do the following:

- Find the coordinates of the vertex
- Find the equation of the Axis of Symmetry
- Find the coordinates of the x-intercept(s)
- Find the coordinates of the y-intercept(s)
- Graph the function.

- Coordinates of the Vertex:

$$x = \frac{-(-2)}{2(1)} = 1$$

$$k(1) = 1^2 - 2(1) + 1 = 0$$

The coordinates are $(1, 0)$

- Equation of the *Axis of Symmetry*:

$$x = 1$$

- Coordinates of the x-intercept(s):

$$x^2 - 2x + 1 = 0$$

$$(x - 1)(x - 1) = 0$$

$$x - 1 = 0 \text{ and } x - 1 = 0$$

then $x = 1$

In this case the x-intercepts are not distinct. Both calculations reveal the same coordinates for the x-intercept $(1, 0)$.

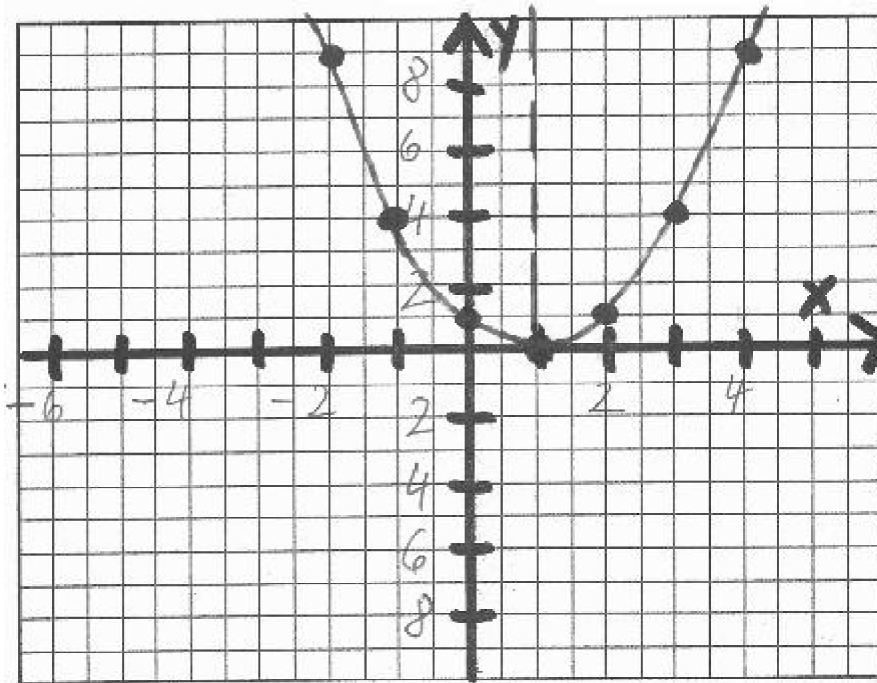
- Coordinates of the y-intercept:

$$k(0) = 0^2 - 2(0) + 1 = 1$$

The coordinates of the y-intercept are $(0, 1)$.

- Now, let's look at the graph:

- The graph is SMOOTH and the vertex is U-shaped.**
- The graph is NEVER parallel to the y-axis. Instead it moves away from it at a steady pace.**



Problem 4:

Given $f(x) = x^2$, do the following:

- Find the coordinates of the vertex
- Find the equation of the Axis of Symmetry
- Find the coordinates of the x-intercept(s)
- Find the coordinates of the y-intercept(s)
- Graph the function.

- Coordinates of the Vertex:

$$x = \frac{-0}{2(1)} = 0$$

$$f(0) = 0^2 = 0$$

The coordinates are $(0, 0)$

- Equation of the *Axis of Symmetry*:

$$x = 0$$

- Coordinates of the x-intercept(s):

$$x^2 = 0$$

$$x = 0 \text{ and } x = 0$$

In this case the x-intercepts are not distinct. Both calculations reveal the same coordinates for the x-intercept $(0, 0)$.

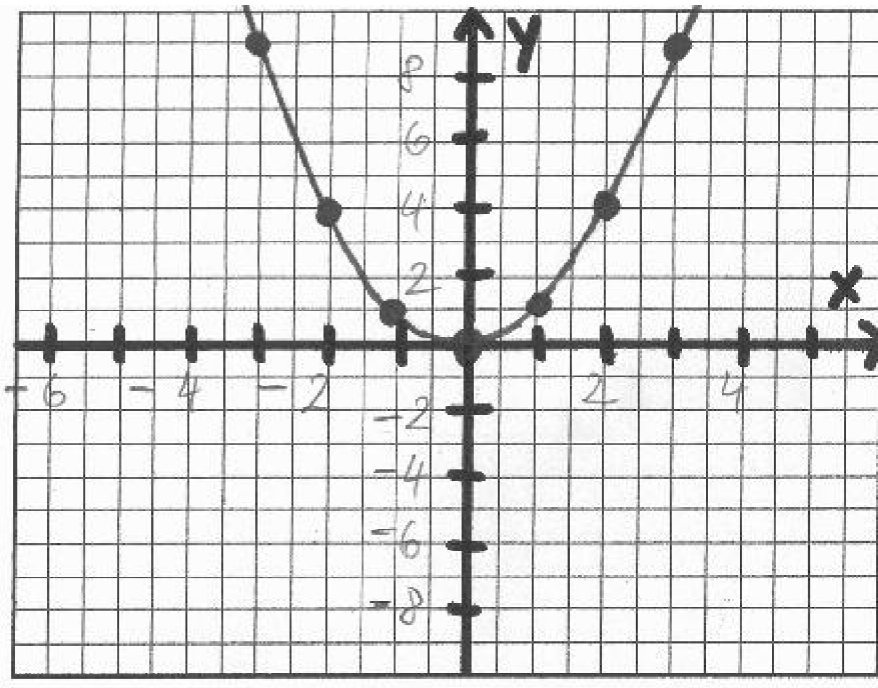
- Coordinates of the y-intercept:

$$f(0) = 0^2 = 0$$

The coordinates of the y-intercept are $(0, 0)$.

Now, let's look at the graph:

1. The graph is **SMOOTH** and the vertex is **U-shaped**.
2. The graph is **NEVER** parallel to the y-axis. Instead it moves away from it at a steady pace.



Problem 5:

Find an equation for the parabola whose vertex is $(1, 2)$ and that passes through the point $(0, 1)$.

Because the parabola has a vertex at $(h, k) = (1, 2)$, the equation must have the form

$$f(x) = a(x - 1)^2 + 2$$

Since the parabola passes through the point $(0, 1)$, we know that $f(0) = 1$.

This allows us to find the value of a as follows:

$$1 = a(0 - 1)^2 + 2$$

$$1 = a + 2$$

$$a = -1$$

which implies that the equation for the parabola is

$$f(x) = -(x - 1)^2 + 2 = -x^2 + 2x + 1$$

Problem 6:

Write the quadratic function $f(x) = -2x^2 + 4x - 8$ in standard form
 $f(x) = a(x - h)^2 + k$.

Use the *Square Completion Method* to change the appearance but not the value of the expression $f(x) = -2x^2 + 4x$.

Next, we will factor out -2 to get $f(x) = -2(x^2 - 2x)$. We now have two of the three terms of a *Perfect Square Trinomial* whose middle term is $-2x$.

Square Completion Method:

1. Divide the coefficient of the middle term by 2 : $\frac{-2}{2}$

2. Raise $\frac{-2}{2}$ to the second power: $\left(\frac{-2}{2}\right)^2 = 1$. We have just found the third term of the *Perfect Square Trinomial*!

3. Insert the third term into the given expression as follows:

$$f(x) = -2(x^2 - 2x + 1) - 8 - (-2)(1)$$

Please note that we actually added -2 to the function inside the parentheses. In order to preserve the value of the function we then had to subtract -2 to preserve the value of the function.

4. Lastly, we factor and combine like terms to get the standard form of the quadratic function

$$f(x) = -2(x - 1)^2 - 6$$

Problem 7:

The monthly profit P , in thousands of dollars, of a company can be estimated by the formula $P(x) = -3x^2 + 30x + 12$, where x is the number of units sold per month. Find the number of units that must be sold by the company to maximize its profit and then find the maximum profit.

$P(x)$ is a quadratic equation whose graph is a parabola open down ($a < 0$). Therefore, this parabola has a vertex that is a maximum. There, the profit is also at a maximum. By finding the x-coordinate of the vertex, we will find the number of units that must be produced to maximize profit. By finding the y-coordinate of the vertex, we will find the maximum profit.

Since the profit function is a quadratic equation in general form, we can use

$$x = \frac{-b}{2a} \text{ to find the x-coordinate of the vertex.}$$

Then, for $a = -3$ and $b = 30$,

$$x = \frac{-b}{2a} = \frac{-30}{2(-3)} = 5, \text{ which is the x-coordinate of the vertex.}$$

The maximum profit occurs when 5 units are sold.

Since $P(5) = -3(5)^2 + 30(5) + 12 = 87$, we have found the y-coordinate of the vertex, which is the maximum profit.

Since the profit is expressed in thousands of dollars, the maximum profit is \$87,000.

Problem 8:

A projectile is shot upward. Its distance above the ground after t seconds is $s(t) = -16t^2 + 400t$. Please note from physics that any object tossed (fired, thrown, shot) into the air follows a parabolic path back to the ground! Calculate the time it takes for the projectile to hit the ground and find the maximum altitude achieved by the projective.

Finding the time it takes for the projective to hit the ground:

In physics, the function s is called the position function. It is assumed that when $s(t) = 0$ the object is on the ground.

Thus, $-16t^2 + 400t = 0$ will give us the time at which the projectile hits the ground.

Let's solve for t by factoring and then using the *Zero Product Principle* as follows:

$$-16t(t - 25) = 0$$

Then

$$-16t = 0$$

$$t = 0$$

and

$$t - 25 = 0$$

$$t = 25$$

This tells us that just before we shot the projectile into the air, that is at $t = 0$, it is on the ground! **From the second calculation we find that the ball is on the ground again after 25 seconds.**

Finding maximum altitude:

Since $s(t) = -16t^2 + 400t$ is a quadratic function, all we have to do is find the y-coordinate of the vertex!

$$t = \frac{-400}{2(-16)} = 12.5$$

$$s(12.5) = -16(12.5)^2 + 400(12.5) = 2500$$

Thus, the maximum altitude of the projectile is 2,500 feet.

Problem 9:

For what values of x is the graph of $f(x) = x^2 - 4x - 5$ below the x-axis?

Since $a > 0$ we know that the graph is open up. Let's find the coordinates of the vertex point!

$$x = -\frac{-4}{2(1)} = 2$$

$$f(2) = 2^2 - 4(2) - 5 = -9$$

The coordinates of the vertex point are $(2, -9)$, which lies below the x-axis.

This means that the graph of the quadratic function for x-values between the x-intercept will lie below the x-axis. Let's find the coordinates of the x-intercepts.

$$0 = x^2 - 4x - 5$$

$$0 = (x - 5)(x + 1)$$

$$x - 5 = 0 \text{ or } x + 1 = 0$$

$$x = 5 \text{ or } x = -1$$

The coordinates for the x-intercepts are $(5,0)$ and $(-1,0)$

Therefore, the graph of $f(x) = x^2 - 4x - 5$ must lie below the x-axis between $x = -1$ and $x = 5$.

Let's look at its graph!

