DETAILED SOLUTIONS AND CONCEPTS - POLYNOMIAL FUNCTIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definition of Polynomial Functions

A polynomial function is an equation in two variables. The right side of the equation consists of a finite number of terms with nonnegative integer exponents on the variable x. The left side of the equation is function notation such as $f(x)$ or simply y. This is usually expressed as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + ...a_2 x^2 + a_1 x + a_0$$

The domain consists of *All Real Numbers*.

Please note the following:

Polynomial functions are always written with their variables arranged in descending order of their power. $a_1\, x$ is assumed to be $a_1\, x^1$ and a_o is assumed to be $a_o\, x^o$ where $x^o = 1$!

Some Definitions

The *degree* of a polynomial function is equal to the highest exponent found on the independent variables.

The *leading coefficient* is the coefficient of the independent variable to highest power.

For example,

$f(x) = x^4 + 3x - 5$ and $f(x) = 3x^5 - 5x + 9$ are polynomial functions of *degree 4* and *degree 5*, respectively. Their *leading coefficients* are 1 and 3, respectively.

$g(x) = -x^3 + 2x^2 - x + 2$ is a polynomial function of *degree 3* and has a *leading coefficient* of -1.

$p(x) = 6x(x-2)(x+1)^2(x+5)^3$ is also a polynomial function, however, it is written as a product of its linear factors. Its *degree* is the sum of the exponents of the linear factors. Its *leading coefficient* is ***6***.

There are two factors, ***6x*** and ***x - 2***, which have degree ***1*** each. There is one factor, ***x + 1***, which has degree ***2***, and a factor, ***x + 5***, which has degree ***3***. When a polynomial function is in factored form, we simply add the degrees of each factor. In our case, we get ***1 +1 + 2 + 3 = 7***.

Examples of Polynomial Functions

- Constant Functions (horizontal lines) are polynomial functions of *degree 0*.
- Linear Functions are polynomial functions of *degree 1*.
- Quadratic Functions are polynomial functions of *degree 2*.
- Any equations in two variables containing a combination of constants and variables with nonnegative exponents are polynomial functions. For example, $f(x) = x^4 + 3x - 5$ or $f(x) = 3x^5 - 5x + 9$ are polynomial functions.

Note that functions of the form $y = x^2 + x^{1/2}$ and $y = x^3 - x^{-2} + 1$ are not polynomials because of a fractional power in the first function and a negative power in the second one.

The *Zeros* of a Polynomial Function

The *Zeros* of a polynomial function are real or imaginary number replacements for ***x*** which give a y-value of ***0***.

According to the *Fundamental Theorem of Algebra*, every polynomial of degree ***n > 0*** has at least one *Zero*.

According to the *n-Zeros Theorem*, every polynomial of degree ***n > 0*** can be expressed as the product of ***n*** linear factors. Hence, a polynomial has exactly ***n*** *Zeros*, not necessarily distinct.

According to the *Factor Theorem*, if a number ***r*** is a *Zero* of the polynomial function, then ***(x - r)*** is a factor of the function and vice versa.

Given the above theorems, we can say the following

- The degree of the polynomial function tells us how many *Zeros* we will get.

- *Zeros* do not have to be distinct.

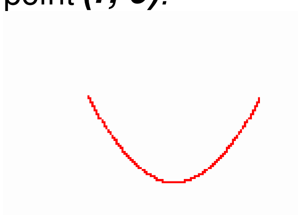
- *Zeros* can be real or imaginary. The real *Zeros* are the x-coordinates of the x-intercepts on the graph of a polynomial function. **Imaginary *Zeros* occur in conjugate *** pairs (Conjugate Pairs Theorem).**

  *** The conjugate of a complex number $a + bi$ is the complex number $a - bi$.

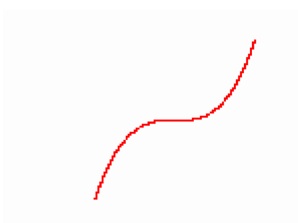
- If a number r is a *Zero* of the polynomial function, then $(x - r)$ is a factor of the function and vice versa.

- If $(x-r)^m$ is a factor of a polynomial function, then r is called a *Zero* of multiplicity m. **Multiplicities help shape the graph of a function.**

  (a) If r is a real *Zero* of even multiplicity, then the graph of the function touches the x-axis at r. Specifically, the graph is parabolic in shape at the point $(r, 0)$.



  (b) If r is a real *Zero* of odd multiplicity greater than 1, then the graph CROSSES the x-axis at r mimicking the picture of a cubic function at the point $(r, 0)$.



  (c) If r is a real *Zero* of multiplicity 1, then the graph CROSSES the x-axis at r in a straight line.

Intermediate Value Theorem

This theorem can show us if a polynomial function actually has a real *Zero* on some interval along the x-axis. Remember that real *Zeros* are the x-coordinates of the x-intercepts!

It states:

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real *Zero* of f between a and b.

Characteristics of Graphs of Polynomial Functions

Unlike linear or quadratic functions, polynomial functions of degree higher than two do not have one standard graph. Infinitely many different graphs are possible.

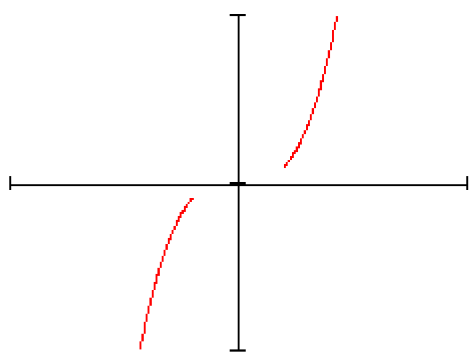
In this course we will only graph polynomial functions with a graphing utility because to do so by hand requires methods that are beyond the scope of this course.

- All polynomial functions of degree higher than two have graphs that consist of continuous curves without breaks.
- The graphs are SMOOTH with rounded turns, and they eventually rise or fall without bound.
- There is always a y-intercept.
- There can be infinitely many x-intercepts, but the graphs of some polynomial functions may not have one.
- The behavior of the graph of a function to the far right and to the far left is called the *end behavior* of a function.

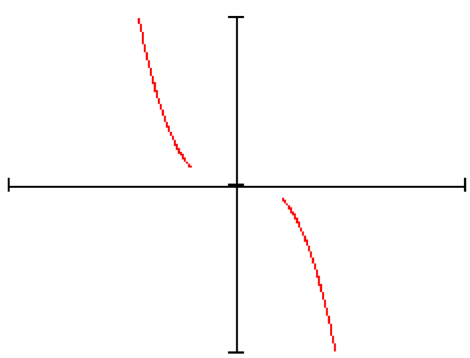
Following are the four types of end behaviors of any polynomial function:

1. When the **degree** of the polynomial is **odd** and the **leading coefficient** is **positive,** the end behavior of the graph is as follows.

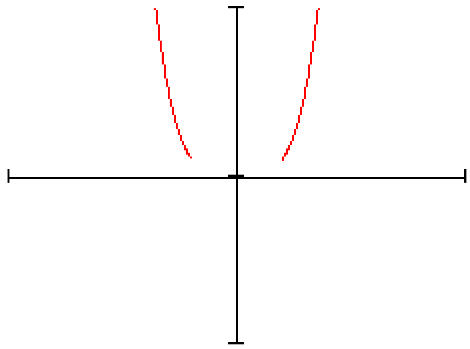
   Note: The behavior in the middle depends on the make-up of the polynomial function. There could be numerous loops in-between the ends.



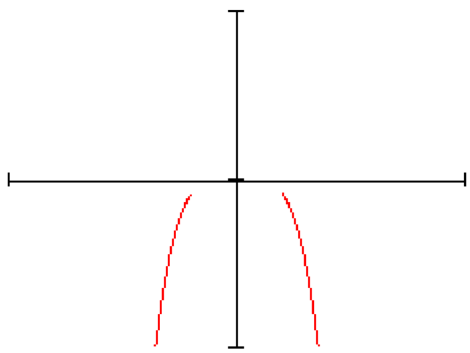
2. When the **degree** of the polynomial is **odd** and the **leading coefficient** is **negative**, the end behavior of the graph is as follows.



3. When the **degree** of the polynomial is **even** and the **leading coefficient** is **positive**, the end behavior of the graph is as follows.

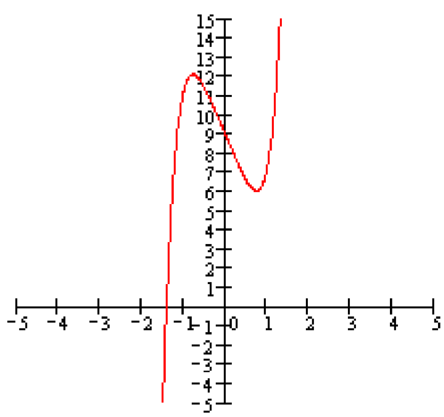


4. When the **degree** of the polynomial is **even** and the **leading coefficient** is **negative**, the end behavior of the graph is as follows.



Problem 1:

Given $f(x) = 3x^5 - 5x + 9$ and its graph



find the following:

a. Leading coefficient and degree of the polynomial
b. Number of real *Zeros* and their approximate values using the graph
c. Number of imaginary *Zeros*

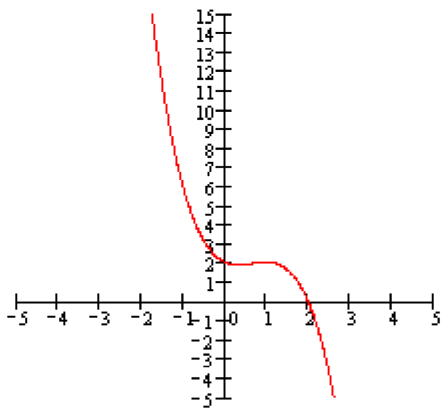
The leading coefficient, ***3***, is positive and the degree of the polynomial is five, which is odd.

We know that the function must have five *Zeros* because it is of degree five. Looking at the graph, we notice that the function has one real *Zero*, which must have an approximate value of ***-1.5*** (the x-intercept).

Since there is only one real *Zero*, the function must have four imaginary *Zeros*. Remember that imaginary *Zeros* occur in conjugate pairs!

Problem 2:

Given $g(x) = -x^3 + 2x^2 - x + 2$ and its graph



find the following:

a. Leading coefficient and degree of the polynomial
b. Number of real *Zero*s and their approximate values using the graph
c. Number of imaginary *Zeros*

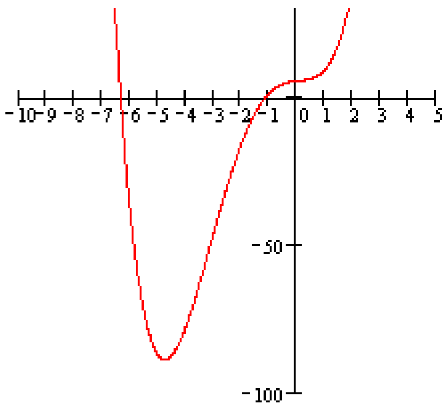
The leading coefficient, ***-1***, is negative and the degree of the polynomial is three, which is odd.

We know that the function must have three *Zeros* because it is of degree three. Looking at the graph, we notice that the function has one real *Zero*, which must have an approximate value of ***2*** (the x-intercept).

Since there is only one real *Zero*, the function must have two imaginary *Zeros*. Remember that imaginary *Zeros* occur in conjugate pairs!

Problem 3:

Given $h(x) = \frac{1}{2}x^4 + 3x^3 - x^2 + x + 5$ and its graph



find the following:

a. Leading coefficient and degree of the polynomial
b. Number of real *Zero*s and their approximate values using the graph
c. Number of imaginary *Zeros*

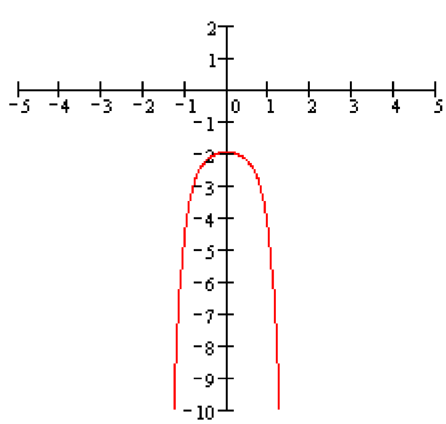
The leading coefficient, $\frac{1}{2}$, is positive and the degree of the polynomial is four, which is even.

We know that the function must have four *Zeros* because it is of degree four. Looking at the graph, we notice that the function has two real *Zeros*, which must have an approximate value of ***-6.25*** and ***-1*** (the x-intercepts).

Since there are two real *Zero*, the function must have two imaginary *Zeros*. Remember that imaginary *Zeros* occur in conjugate pairs!

Problem 4:

Given $k(x) = -\frac{3}{2}x^6 - x^2 - 2$ and its graph



find the following:

a. Leading coefficient and degree of the polynomial
b. Number of real *Zero*s and their approximate values using the graph
c. Number of imaginary *Zeros*

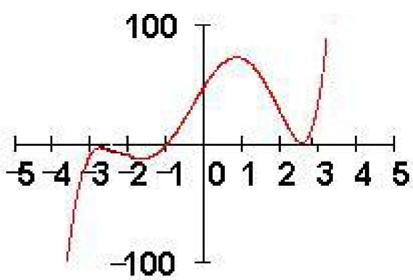
The leading coefficient, $-\frac{3}{2}$, is negative and the degree of the polynomial is six, which is even.

We know that the function must have six *Zeros* because it is of degree six. Looking at the graph, we notice that the function has no real *Zeros* since it has no x-intercepts.

Since there are no real *Zeros*, the function must have six imaginary *Zeros*. Remember that imaginary *Zeros* occur in conjugate pairs!

Problem 5:

Given $g(x) = x^5 + x^4 - 14x^3 - 14x^2 + 49x + 49$ and its graph



find the following:

a. Leading coefficient and degree of the polynomial
b. Number of real *Zero*s and their approximate values using the graph
c. Number of imaginary *Zeros*

The leading coefficient, ***1,*** is positive and the degree of the polynomial is five, which is odd.

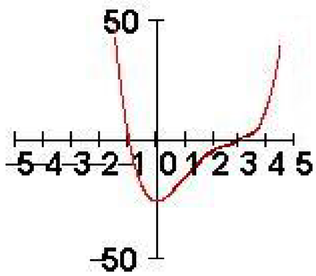
We know that the function must have five *Zeros* because it is of degree five. Looking at the graph, we notice that the function has three real *Zeros*, which must have an approximate value of ***-2.75*** and ***-1*** and ***2.75*** (the x-intercepts). However, since the graph is parabolic in shape at the intercept ***-2.75*** and ***2.75***, they must represent *Zeros* of even multiplicity.

Given a degree of five, we concede that this polynomial must have *double Zeros* at ***-2.75*** and ***2.75*** and a single *Zero* at ***-1***.

Since there are five real *Zeros*, the function has no imaginary *Zeros*.

Problem 6:

Given $m(x) = x^4 - 8x^3 + 18x^2 - 27$ and its graph



find the following:

a. Leading coefficient and degree of the polynomial
b. Number of real *Zero*s and their approximate values using the graph
c. Number of imaginary *Zeros*

The leading coefficient, $\mathbf{1}$ is positive and the degree of the polynomial is four, which is even.

We know that the function must have four *Zeros* because it is of degree four. Looking at the graph, we notice that the function has two real *Zeros*, which must have an approximate value of $\mathbf{-1}$ and $\mathbf{3}$ (the x-intercepts). However, since the graph mimics the picture of a cubic function at the intercept $\mathbf{3}$, it must represent a *Zero* of odd multiplicity greater than 1.

Given a degree of four, we concede that this polynomial must have a *triple Zeros* at $\mathbf{3}$ and a single *Zero* at $\mathbf{-1}$.

Since there are four real *Zeros*, the function has no imaginary *Zeros*.

Problem 7:

Show that $h(x) = \frac{1}{2}x^4 + 3x^3 - x^2 + x + 5$ has a real *Zero* between -2 and -1.

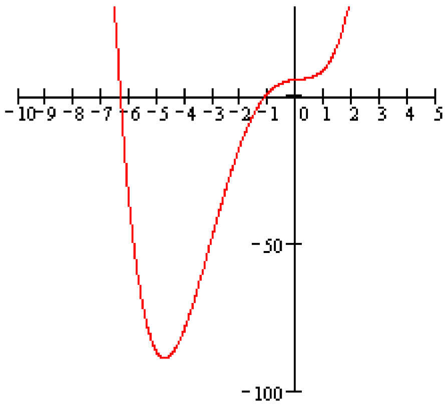
We will use the *Intermediate Value Theorem* to find whether the values of $h(-1)$ and $h(-2)$ are of opposite sign. If this is the case, then by the *Intermediate Value Theorem* we can conclude that there must be an x-intercept, and hence a real *Zero*.

$$h(-1) = \frac{1}{2}(-1)^4 + 3(-1)^3 - (-1)^2 + (-1) + 5 = \frac{1}{2}$$

$$h(-2) = \frac{1}{2}(-2)^4 + 3(-2)^3 - (-2)^2 + (-2) + 5 = -17$$

Since the values of $h(-1)$ and $h(-2)$ are of opposite sign, we can conclude that there is an x-intercept between -2 and -1, and hence a real *Zero*.

Following is the graph of the polynomial to illustrate our reasoning.



Problem 8:

Show that $g(x) = -x^3 + 2x^2 - x + 2$ has a real *Zero* between 1 and 3.

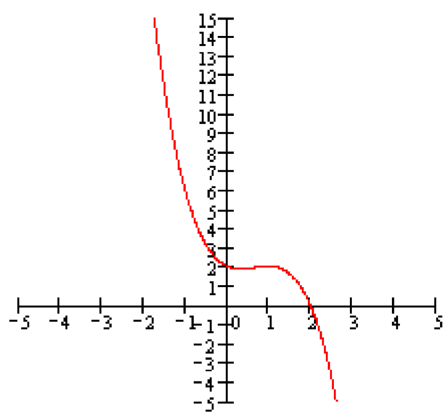
We will use the *Intermediate Value Theorem* to find whether the values of ***g(1)*** and ***g(3)*** are of opposite sign. If this is the case, then by the *Intermediate Value Theorem* we can conclude that there must be an x-intercept, and hence a real *Zero*.

$g(1) = -(1)^3 + 2(1)^2 - 1 + 2 = 2$

$g(3) = -(3)^3 + 2(3)^2 - 3 + 2 = -10$

Since the values of ***g(1)*** and ***g(3)*** are of opposite sign, we can conclude that there is an x-intercept between 1 and 3, and hence a real *Zero*.

Following is the graph of the polynomial to illustrate our reasoning.



Problem 9:

If the polynomial $f(x) = 3x^5 - 5x + 9$ has exactly one real *Zero* ***r*** between -4 and 0, which of the following is true?

-4 < r < -3

-3 < r < -2

-2 < r < -1

-1 < r < 0

We will use the *Intermediate Value Theorem* to find whether the values of ***f(-4)*** and ***f(-3)*** are of opposite sign. If they are not, we will find whether the values of ***f(-3)*** and ***h(-2)*** are of opposite sign, and so on.

$f(-4) = 3(-4)^5 - 5(-4) + 9 = -3043$

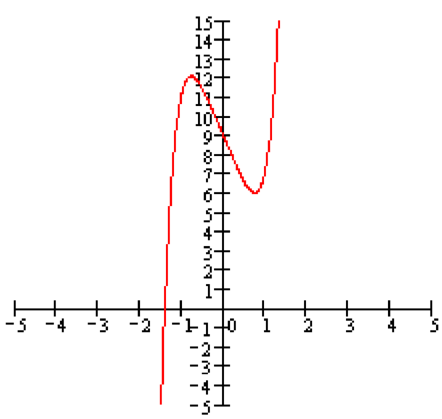
$f(-3) = 3(-3)^5 - 5(-3) + 9 = -705$

$f(-2) = 3(-2)^5 - 5(-2) + 9 = -77$

$f(-1) = 3(-1)^5 - 5(-1) + 9 = 11$

Since the values of ***f(-2)*** and ***f(-1)*** are of opposite sign, we can conclude that there is an x-intercept between -2 and -1, and hence a real *Zero*.

Following is the graph of the polynomial to illustrate our reasoning.



Problem 10:

Find the factored form of a polynomial function with leading coefficient $\boldsymbol{6}$ and the following *Zeros*:

$-\frac{1}{3}$, $\frac{5}{2}$, and $\boldsymbol{3}$

Remember, if a number $\boldsymbol{r}$ is a *Zero* of the polynomial function, then **(x - r)** is a factor of the function.

$$y = 6[x - (-\tfrac{1}{3})](x - \tfrac{5}{2})(x - 3)$$
$$y = 6(x + \tfrac{1}{3})(x - \tfrac{5}{2})(x - 3)$$

Please note, since $6 = 2 \cdot 3$ we can distribute the $\boldsymbol{2}$ to the second factor and the $\boldsymbol{3}$ to the first factor as follows:

$$y = 2 \cdot 3(x + \tfrac{1}{3})(x - \tfrac{5}{2})(x - 3)$$

$$y = (3x + 1)(2x - 5)(x - 3)$$

Problem 11:

Find the factored form of a polynomial function with leading coefficient $\boldsymbol{2}$ and the following *Zeros*:

$1 - \sqrt{3}$, $1 + \sqrt{3}$, and $\frac{2}{3}$ (multiplicity 2)

$$y = 2[x - (1 - \sqrt{3})][x - (1 + \sqrt{3})](x - \tfrac{2}{3})^2$$
$$y = 2(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})(x - \tfrac{2}{3})^2$$

Please note that although two of the *Zeros* for this polynomial function are not combined into one number, we still turn them into factors according to the rule "if a number $\boldsymbol{r}$ is a *Zero* of the polynomial function, then **(x - r)** is a factor of the function.

Problem 12:

Find the factored form of a polynomial function with leading coefficient $\boldsymbol{2}$ and the following *Zeros*:

$1 - 2i$, $1 + 2i$, and $\boldsymbol{-5}$ (multiplicity 3)

$$y = 2[x - (1 - 2i)][x - (1 + 2i)][x - (-5)]^3$$
$$y = 2(x - 1 + 2i)(x - 1 - 2i)(x + 5)^3$$

Problem 13:

Find the factored form of a polynomial function with leading coefficient ***1*** and the following *Zeros*:

-5 (multiplicity 3), $-2 - i\sqrt{7}$, and $-2 + i\sqrt{7}$

Please note that it is customary and standard to place the imaginary number *i* in front of radicals instead of after them as is done with rational numbers. See Example 9 above!

$$p(x) = [x - (-5)]^3[x - (-2 - i\sqrt{7})][x - (-2 + i\sqrt{7})]$$

and

$$p(x) = (x + 5)^3(x + 2 + i\sqrt{7})(x + 2 - i\sqrt{7})$$

Please note that although two of the *Zeros* for this polynomial function are not combined into one number, we still turn them into factors according to the rule "if a number ***r*** is a *Zero* of the polynomial function, then ***(x - r)*** is a factor of the function.

Problem 14:

Find the factored form of three polynomial functions with the following *Zeros* and leading coefficients ***1***, ***5***, and ***-10***.

0, ***2***, and ***-3*** (multiplicity 2)

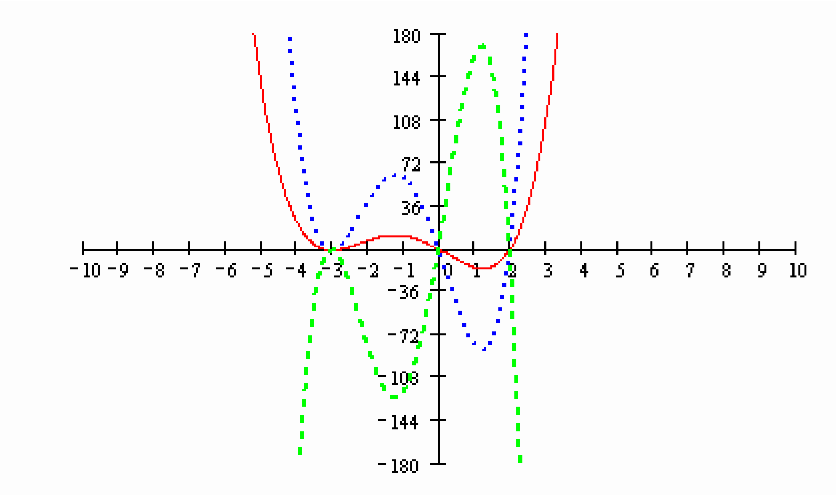
$$p(x) = x(x - 2)(x + 3)^2$$

$$f(x) = 5x(x - 2)(x + 3)^2$$

$$g(x) = -10x(x - 2)(x + 3)^2$$

Following is a graph showing all three functions. Note that these functions have the same *Zeros*. The different leading coefficients affect the location of the peaks and valleys.

The solid graph (red) shows the function ***p*** ; the dotted graph (blue) shows the function ***f*** ; and the dashed graph (green) shows the function ***g***.



Problem 15:

If ***i*** is a *Zero*, what other imaginary number MUST also be a *Zero* according to the *Conjugate Pairs Theorem*?

Answer: ***-i*** because imaginary *Zeros* occur in conjugate pairs.

Problem 16:

If ***3 - 2i*** is a *Zero*, what other imaginary number MUST also be a *Zero* according to the *Conjugate Pairs Theorem*?

Answer: ***3 + 2i*** because imaginary *Zeros* occur in conjugate pairs.

Problem 17:

If ***1 + 6i*** is a *Zero*, what other imaginary number MUST also be a *Zero* according to the *Conjugate Pairs Theorem*?

Answer: ***1 - 6i*** because imaginary *Zeros* occur in conjugate pairs.

Problem 18:

If ***-5 - 6i*** is a *Zero*, what other imaginary number MUST also be a *Zero* according to the *Conjugate Pairs Theorem*?

Answer: ***-5 + 6i*** because imaginary *Zeros* occur in conjugate pairs.