DETAILED SOLUTIONS AND CONCEPTS - PIECEWISE-DEFINED FUNCTIONS

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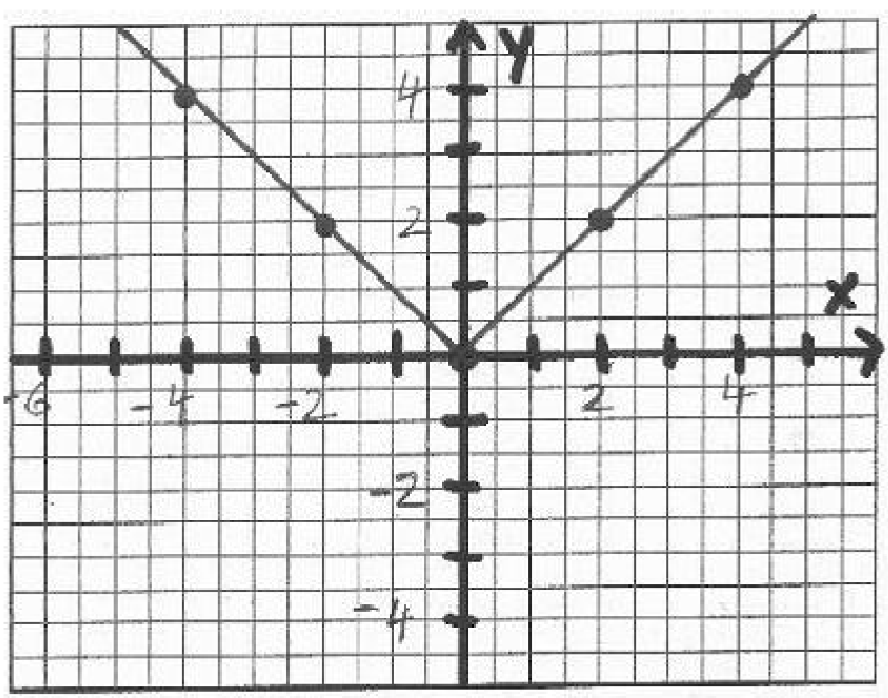
PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Sometimes, data is modeled by two or more equations, each with its own domain restriction. When these equations are written in function notation, such functions are called piecewise-defined functions.

The equations making up a piecewise-defined functions may be linear or nonlinear. Actually, you already know one piecewise-defined function, namely the absolute value function $f(x) = |x|$.

Check out its graph! Each of its branches is a linear function!



This function is often represented as follows:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

, where x is the line representing the right branch, and $-x$ is the line representing the left branch!

$x \geq 0$ indicates the domain for the branch x and $x < 0$ indicates the domain for the branch $-x$.

Characteristics of the Graphs of Piecewise-Defined Functions

- The graphs of some functions may not be a smooth curve.
- The graphs of some functions may have discontinuities such as holes or "jumps". Holes are indicated with "open" circles —o.
- If a graph ends at a distinct point, this is indicated with a "closed" circle —●.
- The graphs of some functions may not have any intercepts.
- A graph may have at most one y-intercept.
- There can be many x-intercepts.

Strategy for Graphing Piecewise-Defined Functions

- Create a *Cartesian Coordinate System* as discussed in Lecture 01.
- Find the following for each branch separately keeping in mind the characteristics of its graph:

  a. the intercepts, if they exist.
  b. the vertex, if it exists.
  c. the point at which concavity changes, if it exists.
  d. the point at which the function starts, if it exists.
  e. all holes, if there are any.

- Find and plot at least 5 other points to better facilitate the shape of the graph, particularly the concavities.
- Connect all points found in the previous steps keeping in mind the shape of the graph and the discontinuities.

Problem 1:

The function f is defined as

$$f(x) = \begin{cases} x+3 & \text{if } x \leq 0 \\ 3 & \text{if } 0 < x < 2 \\ 2x-1 & \text{if } x > 2 \end{cases}$$

(a) Find $f(0)$, $f(1)$, $f(2)$, and $f(3)$.

Since $x = 0$ is only in the domain of the branch $y = x + 3$, then $f(0) = 0 + 3 = 3$.

Since $x = 1$ is only in the domain of the branch $y = 3$, then $f(1) = 3$.

Since $x = 2$ is NOT in the domain of any of the branches, there is NO value for $f(2)$.

Since $x = 3$ is only in the domain of the branch $y = 2x - 1$, then $f(3) = 2(3) - 1 = 5$

(b) Determine the coordinates of the intercepts of the function.

Coordinates of the x-intercept(s):

In the case of piecewise-defined functions, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = x + 3$, we find the x-intercept as follows:

$$0 = x + 3$$
$$x = -3$$

Since -3 is in the domain of this branch, the coordinates of the x-intercept are $(-3,0)$.

For the branch $y = 3$, we can rule out an x-intercept immediately. Since the y-value is always 3 and never 0.

For the branch $y = 2x - 1$, we find the x-intercept as follows:

$$0 = 2x - 1$$
$$x = \tfrac{1}{2}$$

Since $\frac{1}{2}$ is **NOT** in the domain of this branch, it does NOT produce any x-intercepts.

Coordinates of y- intercept:

Again, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = x + 3$, we find

$$y = 0 + 3$$

$$y = 3$$

Since $x = 0$ is in the domain of this branch, the coordinates of the y-intercept are $(0,3)$

For the branches $y = 3$ and $y = 2x - 1$, we find that $x = 0$ is **NOT** in their domain. Therefore, these branches do **NOT** produce any y-intercepts.

(c) Graph the function.

When you graph a piecewise-defined function, you must graph all of the branches separately using the domains specified.

In our case we have the following branches with their domains:

Branch 1: $y = x + 3$ with domain $x \leq 0$

Using the *Point-by-Point Plotting Method*, let's find some points. We may as well just pick integers for x! Note that we cannot pick any positive values for x due to the domain restriction!

x	-5	-4	-3	-2	0
y	-2	-1	0	1	3

Branch 2: $y = 3$ with domain $0 < x < 2$.

Notice, although 0 and 2 are **NOT** in the domain of this branch, we use them as a **placeholder** for the first number after 0 and the last number before 2 in the domain.

x	0	1	2
y	3	3	3

Branch 3: $y = 2x - 1$ with domain $x > 2$.

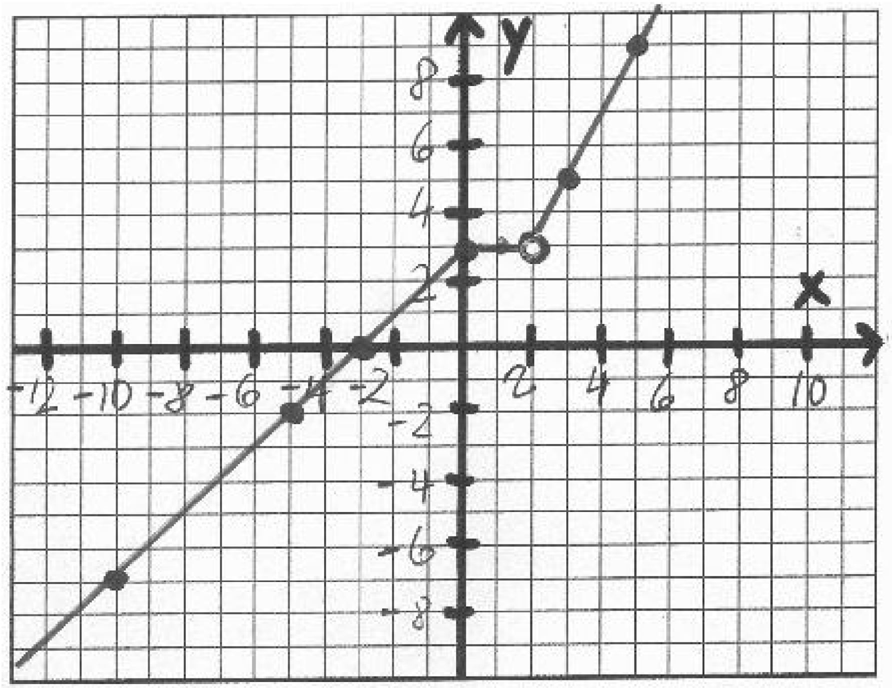
Notice, although 2 is **NOT** in the domain of this branch, we use it as a **placeholder** for the first number after 2 in the domain.

x	2	3	4	5
y	3	5	7	9

Below is the graph of the function.

Please note that although 0 is **NOT** in the domain of Branch 2, it is in the domain of Branch 1. This is indicated with a solid point on the graph.

However, 2 is neither in the domain of Branch 2 nor in the domain of Branch 3. This is indicated with a circle on the graph, which is considered a "hole."



Please observe the SCALE of the Coordinate System! Note that the units along the x- and y-axis are the SAME when we graph lines.

Problem 2:

The function g is defined as

$$g(x) = \begin{cases} 3x - 1 & \text{if } x < 2 \\ -x + 3 & \text{if } x > 4 \end{cases}$$

(a) Find $g(0)$, $g(4)$, and $g(5)$.

Since $x = 0$ is only in the domain of the branch $y = 3x - 1$, then $g(0) = 3(0) - 1 = -1$.

Since $x = 4$ is **NOT** in the domain of any of the branches, there is NO value for $g(4)$.

Since $x = 5$ is only in the domain of the branch $y = -x + 3$, then $g(5) = -(5) + 3 = -2$.

(b) Determine the coordinates of the intercepts of the function.

Coordinates of the x-intercept(s):

In the case of piecewise-defined functions, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = 3x - 1$, we find the x-intercept as follows:

$$0 = 3x - 1$$
$$1 = 3x$$
$$x = \frac{1}{3}$$

Since $\frac{1}{3}$ is in the domain of this branch, the coordinates of the x-intercept are $(\frac{1}{3}, 0)$

For the branch $y = -x + 3$, we find

$$0 = -x + 3$$

$$x = 3$$

Since 3 is **NOT** in the domain of this branch, it does **NOT** produce any x-intercepts.

Coordinates of y- intercept:

Again, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = 3x - 1$, we find

$$y = 3(0) - 1$$
$$y = -1$$

Since 0 is in the domain of this branch, the coordinates of the y-intercept are $(0, -1)$

For the branch $y = -x + 3$, we find that $x = 0$ is **NOT** in its domain. Therefore, this branch does **NOT** produce any y-intercepts.

(c) Graph the function.

We have the following branches with their domains:

Branch 1: $y = 3x - 1$ with domain $x < 2$

Notice, although 2 is **NOT** in the domain of this branch, we use it as a **placeholder** for the last number before 2 in the domain.

x	-1	0	1	2
y	-4	-1	2	5

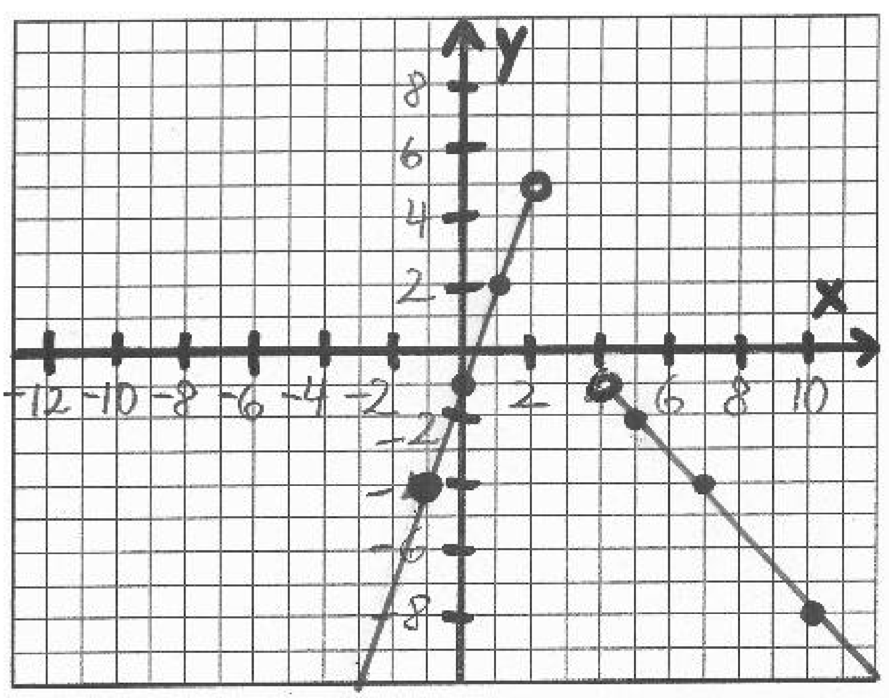
Branch 2: $y = -x + 3$ with domain $x > 4$.

Notice, although 4 is **NOT** in the domain of this branch, we use it as a **placeholder** for the first number after 4 in the domain.

x	4	5	6	7
y	-1	-2	-3	-4

Below is the graph of the function.

Please note that 2 and 4 are **NOT** in the domain of either branch. This is indicated with circles, which are considered "holes."



Please observe the SCALE of the Coordinate System! Note that the units along the x- and y-axis are the SAME when we graph lines.

Problem 3:

The function h is defined as

$$h(x) = \begin{cases} \frac{5}{4}x + \frac{5}{4} & \text{if } x < 3 \\ -3x + 12 & \text{if } x \geq 3 \end{cases}$$

(a) Determine the coordinates of the intercepts of the function.

Coordinates of the x-intercept(s):

In the case of piecewise-defined functions, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = \frac{5}{4}x + \frac{5}{4}$, we find

$$0 = \frac{5}{4}x + \frac{5}{4}$$
$$-\frac{5}{4} = \frac{5}{4}x$$
$$x = -1$$

Since -1 is in the domain of this branch, the coordinates of the x-intercept are $(-1, 0)$.

For the branch $y = -3x + 12$, we find the x-intercept as follows:

$$0 = -3x + 12$$
$$-12 = -3x$$
$$x = 4$$

Since 4 is in the domain of this branch, the coordinates of the x-intercept are $(4, 0)$.

Coordinates of y- intercept:

Again, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = \frac{5}{4}x + \frac{5}{4}$, we find

$$y = \frac{5}{4}(0) + \frac{5}{4}$$
$$y = \frac{5}{4}$$

Since $x = 0$ is in the domain of this branch, the coordinates of the y-intercept are $(0, \frac{5}{4})$.

For the branch $y = -3x + 12$, we find that $x = 0$ is **NOT** in its domain. Therefore, this branch does **NOT** produce any y-intercepts.

(b) Graph the function.

We have the following branches with their domains:

Branch 1: $y = \frac{5}{4}x + \frac{5}{4}$ with domain $x < 3$.

Notice, although 3 is **NOT** in the domain of this branch, we use it as a **placeholder** for the last number before 3 in the domain.

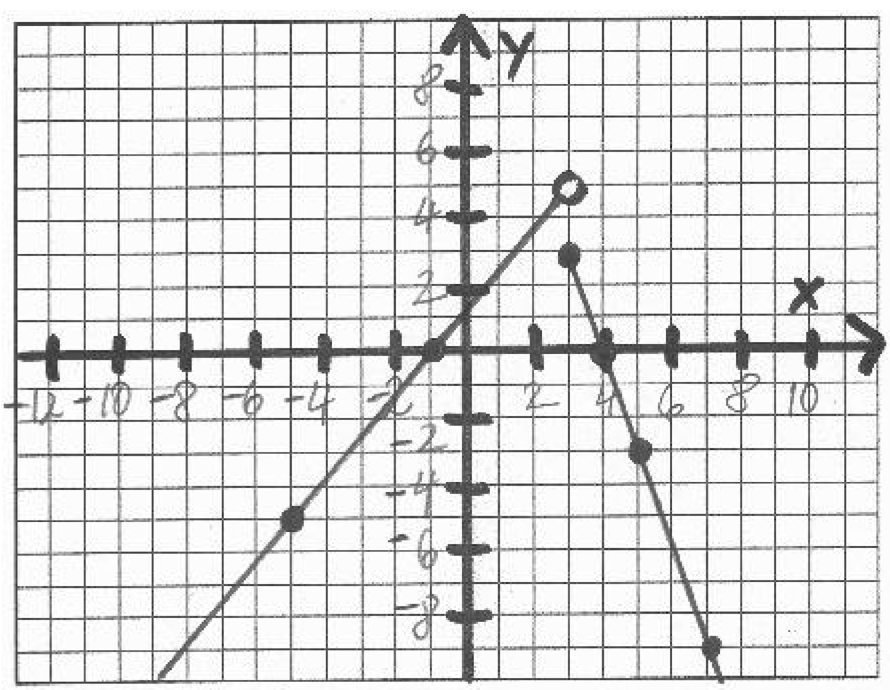
x	-3	0	2	3
y	$-\frac{10}{4}$	$\frac{5}{4}$	$\frac{15}{4}$	5

Branch 2: $y = -3x + 12$ with domain $x \geq 3$.

x	3	5	6
y	3	-3	-6

Below is the graph of the function.

Please note that 3 is **NOT** in the domain of Branch 1, however, it is in the domain of Branch 2. This is indicated with a circle and a dot, respectively. The circle is considered to be a "hole."



Please observe the SCALE of the Coordinate System! Note that the units along the x- and y-axis are the SAME when we graph lines.

Problem 4:

The function ***f*** is defined as

$$f(x) = \begin{cases} 2x+1 & \text{if } x > 2 \\ 2x+1 & \text{if } x < 2 \end{cases}$$

(a) Determine the coordinates of the intercepts of the function.

Coordinates of the x-intercept(s):

In the case of piecewise-defined functions, we need to examine all branches and then rule out the solution that is not in the domain of a branch

In this case, both branches only differ in their domain restrictions.

$$0 = 2x + 1$$

$$-1 = 2x$$

$$x = -\tfrac{1}{2}$$

Since $-\frac{1}{2}$ is in the domain of this branch, the coordinates of the x-intercept are $(-\frac{1}{2}, 0)$.

Coordinates of y- intercept:

Both branches are the same! Therefore,

$$y = 2(0) + 1$$

$$y = 1$$

Since $x = 0$ is in the domain of both branches, the coordinates of the y-intercept are ***(0, 1)***.

(b) Graph the function.

We have the following branches with their domains:

Branch 1: ***y = 2x + 1*** with domain ***x > 2***.

Notice, although ***2*** is **NOT** in the domain of this branch, we use it as a **placeholder** for the last number before ***2*** in the domain.

x	*2*	*4*
y	*5*	*9*

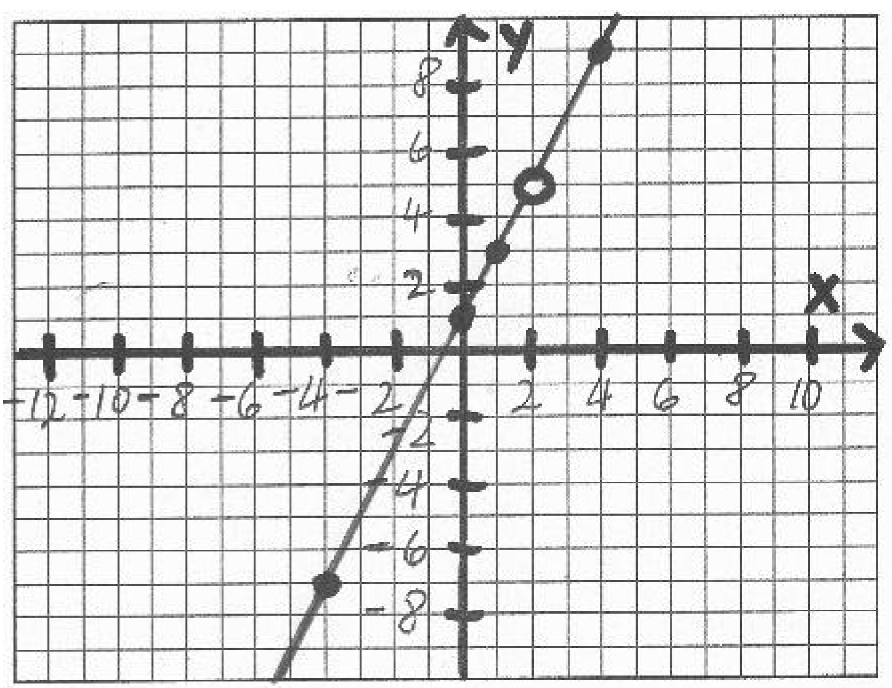
Branch 2: ***y = 2x + 1*** with domain ***x < 2***.

Notice, although ***2*** is **NOT** in the domain of this branch, we use it as a **placeholder** for the last number before ***2*** in the domain.

x	*-4*	*1*	*2*
y	*-7*	*3*	*5*

Below is the graph of the function.

Please note that ***2*** is neither in the domain of Branch 1 nor in the domain of Branch 2. This is indicated with a circle, which is considered a "hole."



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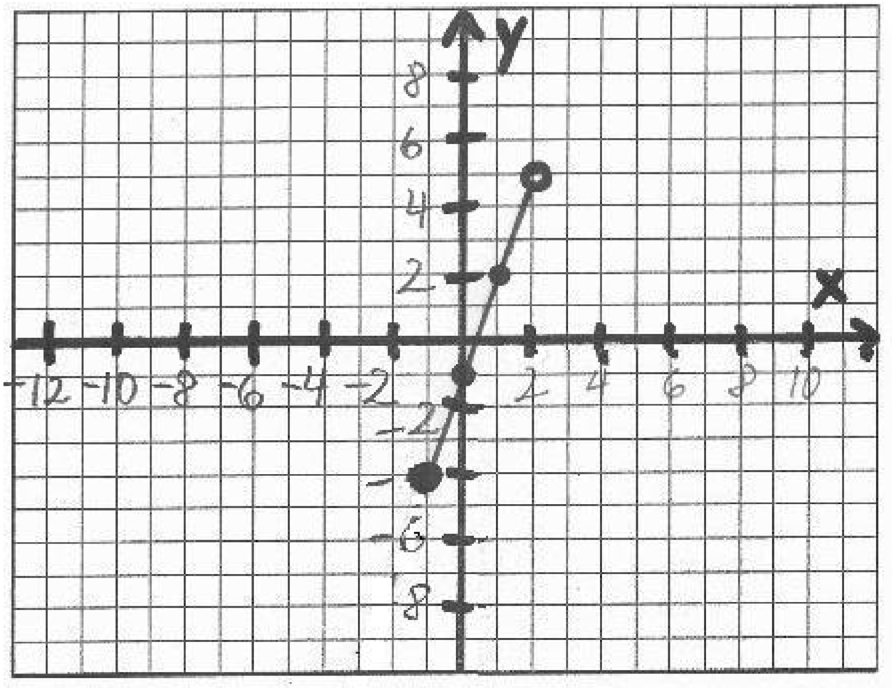
Problem 5:

Find the domain and range of the following function. Write them in *Interval Notation*.



Looking at the graph, the domain must consist of numbers between -1 and 2 along the x-axis. Given a solid dot, the number -1 must be included in the domain whereas the number 2 is not included since there the ending point is a circle.

Therefore, the domain is ***[-1, 2)***.

On the other hand, the range must consist of numbers between -4 and 5 along the y-axis. Given a solid dot, the number -4 must be included in the range whereas the number 5 is not included since there the ending point is a circle.

Therefore, the range is ***[-4, 5)***.

Problem 6:

The function g is defined as

$$g(x)=\begin{cases}\sqrt{x+1} & \text{if } x\geq -1\\ |x+1| & \text{if } x<-1\end{cases}$$

(a) Determine the coordinates of the intercepts of the function.

Coordinates of the x-intercept(s):

In the case of piecewise-defined functions, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = \sqrt{x+1}$, we find

$$
\begin{aligned}
0 &= \sqrt{x+1} \\
0 &= x+1 \\
x &= -1
\end{aligned}
$$

Since -1 is in the domain of this branch, the coordinates of the x-intercept are ***(-1, 0)***.

For the branch $y = |x+1|$, we find the x-intercept as follows:

$$0 = |x+1|$$

In this case, we have to solve two equations.

$0 = x + 1$ or $0 = -(x + 1)$

then $x = -1$

or $0 = -x - 1$ and $x = -1$

In either case, $x = -1$, which is not in the domain of the branch. However, as we have seen above, it is in the domain of the other branch.

Coordinates of y- intercept:

Again, we need to examine all branches and then rule out the solution that is not in the domain of a branch.

For the branch $y = \sqrt{x+1}$, we find that $x = 0$ is **NOT** in its domain. Therefore, this branch does **NOT** produce any y-intercepts.

For the branch $y = |x+1|$, we find that

$$
\begin{aligned}
y &= |0+1| \\
&= |1| \\
&= 1
\end{aligned}
$$

Since 0 is in the domain of this branch, the coordinates of the y-intercept are ***(0, 1)***.

(b) Graph the function.

We have the following branches with their domains:

Branch 1: $y = \sqrt{x+1}$, domain $x \geq -1$

Notice that this is a transformation of the function $y = \sqrt{x}$ of ***1*** unit to the left.

Therefore, its graph starts at ***(-1, 0)***. Given the domain restriction $x \geq -1$, this starting point is included in the graph.

Other points lying on this branch are as follows:

x	0	3	8
y	1	2	3

Branch 2: $y = |x+1|$, domain $x < -1$.

Notice that this is a transformation of the function $y = |x|$ of ***1*** unit to the left.

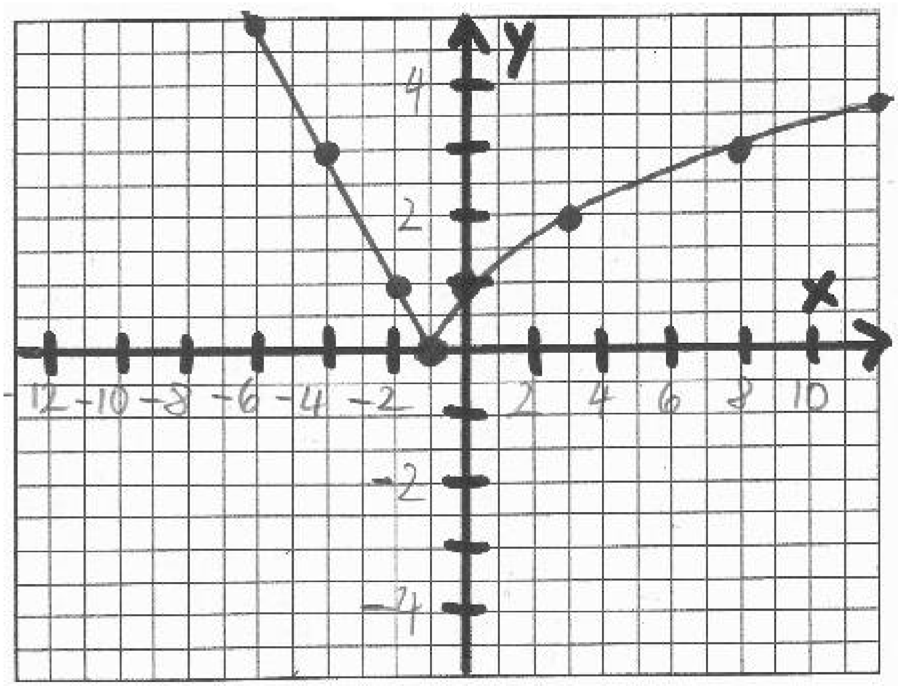
Therefore, its graph has a cusp at ***(-1, 0)***. Given the domain restriction $x < -1$, the cusp is **NOT** included in the graph.

Notice, although ***-1*** is **NOT** in the domain of this branch, we use it as a **placeholder** for the last number before ***-1*** in the domain.

x	-4	-3	-2	-1
y	3	2	1	0

Below is the graph of the function.

Please note that while the point ***(-1, 0)*** is not part of Branch 2, it is, however, part of Branch 1. Therefore, there is **NO** hole in the graph.



Please observe the SCALE of the Coordinate System! Note that the units along the x-axis are DIFFERENT from the units along the y-axis! As long as you place numbers along your axes it does not matter "how long" your units are!