DETAILED SOLUTIONS AND CONCEPTS - SYSTEMS OF NON-LINEAR EQUATIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

All systems of equations could be solved graphically, but this method tends to give us inaccurate results. Therefore, it is NOT an acceptable solution method. **Instead, we MUST solve systems of equations using the *Substitution* or the *Addition Method*.**

The *Substitution Method* works extremely well for finding solutions of *Systems of Linear and Non-linear Equations.* The *Addition Method* is often used for linear system, but cannot always be used for non-linear systems.

Strategy for Solving Systems of Equations by the Substitution Method

Step 1: Solve any one of the equations for one variable in terms of the other. If one of the equations is already in this form, you can skip this step.

Step 2: Substitute the expression found in Step 1 into the other equation. You should now have an equation in one variable. Find its value.

Please note that ONLY at the point of intersection two equations are equal to each other. By setting the x- or y-value of one equation equal to the x- or y-value of the other equation, we are in effect finding the point of intersection.

Step 3: To find the value of the second variable, back-substitute the value of the variable found in Step 2 into one of the original equations. If one of your equations is non-linear, you must back-substitute into both equations and rule out "extraneous solutions."

Step 4: Form an ordered pair with the values found in Step 3 and Step 4. This is the solution to your system of equations.

Strategy for Solving Systems of Equations by the Addition Method

Step 1: If necessary, rewrite both equations in the same form so that the variables and the constants match up when one equation is beneath the other.

Step 2: If necessary, multiply either equation or both equations by appropriate numbers so that the coefficient of either x or y will be opposite in sign giving a sum of 0.

Step 3: Write the equations one below the other, draw a horizontal line, then add each of their terms. The sum should be an equation in one variable. Find its value.

Step 4: To find the value of the second variable, back-substitute the value of the variable found in Step 3 into one of the original equations. If one of your equations is non-linear, you must back-substitute into both equations and rule out "extraneous solutions."

Step 5: Form an ordered pair with the values found in Step 4 and Step 5. This is the solution to your system of equations.

Problem 1:

Solve the following system. Express your answer(s) as coordinates.

$$y - x^2 = -11$$

$$y^2 + x^2 = 13$$

This system contains a quadratic equation and a circle. We will use the *Substitution Method*, which means that we have to solve one of the equations for either x or y.

Step 1:

Using $y - x^2 = -11$, we find

$$y = -11 + x^2$$

Step 2:

Next, we will back-substitute into the equation $y^2 + x^2 = 13$ to get

$$(-11 + x^2)^2 + x^2 = 13$$

$$121 - 22x^2 + x^4 + x^2 = 13$$

$$x^4 - 21x^2 + 108 = 0$$

$$(x^2 - 12)(x^2 - 9) = 0$$

Using the *Zero Property Principle* we get

$x^2 - 12 = 0$ or $x^2 - 9 = 0$

and $x = \pm\sqrt{12} = \pm 2\sqrt{3}$ or $x = \pm 3$

Therefore, the x-coordinates for the solutions are

$x = 2\sqrt{3}$, $x = -2\sqrt{3}$, $x = 3$, and $x = -3$

Step 3:

Since both equations are non-linear, we must find the values for y using both original equations.

Using the quadratic equation $y - x^2 = -11$:

- If $x = 2\sqrt{3}$, then $y - (2\sqrt{3})^2 = -11$

  $y - 12 = -11$

  $y = 1$

  Thus, one solution is $(2\sqrt{3}, 1)$

- If $x = -2\sqrt{3}$, then $y - (-2\sqrt{3})^2 = -11$

  $y - 12 = -11$

  $y = 1$

  and $(-2\sqrt{3}, 1)$ **is another solution**

- If $x = 3$, then $y - (3)^2 = -11$

  $y - 9 = -11$

  $y = -2$

  and *(3, -2)* is yet another solution

- If $x = -3$, then $y - (-3)^2 = -11$

  $y - 9 = -11$

  $y = -2$

  thus, *(-3, -2)* is the last solution

Using the circle $y^2 + x^2 = 13$:

- If $x = 2\sqrt{3}$, then $y^2 + (2\sqrt{3})^2 = 13$

  $y^2 + 12 = 13$
  $y^2 = 1$
  $y = \pm 1$

  Thus, one solution is $(2\sqrt{3}, 1)$, but there is another solution, namely, $(2\sqrt{3}, -1)$. This solution did not show up when we used the quadratic equation, therefore, it must be an extraneous solution. It can be shown that when using the other x-values we will get extraneous solutions as well.

Therefore, the solutions to the given system are $(2\sqrt{3}, 1)$, $(-2\sqrt{3}, 1)$, ***(3, -2)***, and ***(-3, -2)***.

Problem 2:

Solve the following system. Express your answer(s) as coordinates.

$x + y = 0$

$x^3 - 5x - y = 0$

This system contains a linear and a polynomial equation. Let's use the *Substitution Method* and solve the first equation for x.

$x = -y$

then $(-y)^3 - 5(-y) - y = 0$

$-y^3 + 4y = 0$

and $y(-y^2 + 4) = 0$

Using the *Zero Property Principle* we get

then $y = 0$ and $-y^2 + 4 = 0$.

Using the *Square Root Property* we find

$$-y^2 + 4 = 0$$
$$y^2 = 4$$
$$y = \pm\sqrt{4} = \pm 2$$

which means that $y = 2$ and $y = -2$.

Since NOT both equations are linear, we must find the values for y using both original equations.

Using the linear equation $x + y = 0$:

- If $y = 0$, then $x + 0 = 0$

  Thus, one solution is $(0,0)$

- If $y = 2$, then $x + 2 = 0$

  and the second solution is $(-2,2)$

- If $y = -2$, then $x - 2 = 0$

  Thus, one solution is $(2,-2)$

Using the polynomial equation $x^3 - 5x - y = 0$:

- If $y = 0$, then $x^3 - 5x - (0) = 0$

  $$x(x^2 - 5) = 0$$

  and $x = 0$ and

  $$x^2 - 5 = 0$$
  $$x = \pm\sqrt{5}$$

Thus, one solution is $(0,0)$, but there are two other solutions, namely, $(-\sqrt{5},0)$ and $(\sqrt{5},0)$. These solutions did not show up when we used the linear equation, therefore, they must be extraneous solutions. It can be shown that when using the other x-values we will get extraneous solutions as well.

Therefore, the solutions to the given system are $(0,0)$, $(-2,2)$, and $(2,-2)$.

Problem 3:

Solve the following system. Express your answer(s) as coordinates.

$$x^2 + y^2 = 1$$

$$y = -x + 3$$

This system contains a linear equation and a circle. We will use the *Substitution Method*, which means, that we have to solve one of the equations for either x or y. In this example, $y = -x + 3$ is already solved for y.

Next, we will back-substitute into the equation $x^2 + y^2 = 1$

$$x^2 + (-x + 3)^2 = 1$$

$$x^2 + (x^2 - 6x + 9) = 1$$

$$2x^2 - 6x + 8 = 0.$$

This is a quadratic equation that is not factorable. Therefore, we have to use the *Quadratic Formula* to solve for x.

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(2)(8)}}{2(2)} = \frac{6 \pm \sqrt{-28}}{4}$$

$$x = \frac{6 \pm 2\sqrt{-7}}{4} = \frac{2(3 \pm i\sqrt{7})}{4}$$

$$x = \frac{3 + i\sqrt{7}}{2} \text{ or } x = \frac{3 - i\sqrt{7}}{2}$$

The x-coordinates for the points of intersection are imaginary numbers.

Therefore, there are NO solutions to this system. The two graphs do not intersect.

Problem 4:

Solve the following system. Express your answer(s) as coordinates.

$$y = x^2 - 2x$$

$$y = x - 2$$

This system contains a linear and a quadratic equation. In this case, both equations are already solved for ***y***. Therefore, we will definitely use the *Substitution Method*.

$x - 2 = x^2 - 2x$ Notice, that we are simply setting the two equations equal to each other!

$$0 = x^2 - 3x + 2$$

$$0 = (x-2)(x-1)$$

Using the *Zero Property Principle* we get

$x - 2 = 0$ or $x - 1 = 0$

and $x = 2$ and $x = 1$

Since not both equations are linear, we must find the values for ***y*** using both original equations.

Using the linear equation $y = x - 2$:

- If $x = 2$, then $y = 2 - 2 = 0$

  Thus, one solution is *(2, 0)*.

- If $x = 1$, then $y = 1 - 2 = -1$

  and the second solution is *(1, -1)*.

Using the quadratic equation $y = x^2 - 2x$:

- If $x = 2$, then $y = (2)^2 - 2(2) = 0$

  Thus, just like in the case of the linear equation, the quadratic equation produces exactly the same solution, namely ***(2, 0)***. It can be shown that when using the other x-value we will get extraneous solutions as well.

Therefore, the solutions to the given system are ***(2, 0)*** and ***(1, -1)***.

Problem 5:

Solve the following system. Express your answer(s) as coordinates.

$$y = 2x$$

$$xy = 4$$ NOTE: $y = \frac{4}{x}$

This system contains a linear and a rational equation. Let's use the *Substitution Method* since the first equation is already solved for y. Therefore, we can back-substitute into the second equation as follows

$$x(2x) = 4$$
$$2x^2 = 4$$

Using the *Square Root Property* we find

$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

which means that $x = \sqrt{2}$ and $x = -\sqrt{2}$

Since NOT both equations are linear, we must find the values for y using both original equations.

Using the linear equation $y = 2x$:

- If $x = \sqrt{2}$, then $y = 2\sqrt{2}$

  Thus, one solution is $(\sqrt{2}, 2\sqrt{2})$.

- If $x = -\sqrt{2}$, then $y = -2\sqrt{2}$

  and the second solution is $(-\sqrt{2}, -2\sqrt{2})$.

Using the rational equation $xy = 4$:

- If $x = \sqrt{2}$, then $\sqrt{2}y = 4$

  and $y = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

Thus, just like in the case of the linear equation, the rational equation produces exactly the same solution, namely $(\sqrt{2}, 2\sqrt{2})$. It can be shown that when using the other x-value we will get extraneous solutions as well.

Therefore, the solutions to the given system are $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$.

Problem 6:

Solve the following system. Express your answer(s) as coordinates.

$$y = x - 3$$

$$x^2 + y^2 = 9$$

This system contains a linear equation and a circle. We will use the *Substitution Method*, which means, that we have to solve one of the equations for either x or y. In this example, $y = x - 3$ is already solved for y.

Next, we will back-substitute into the equation $x^2 + y^2 = 9$

$$x^2 + (x - 3)^2 = 9$$

$$x^2 + (x^2 - 6x + 9) = 9$$

$$x^2 + x^2 - 6x + 9 = 9$$
$$2x^2 - 6x = 0$$
$$2x(x - 3) = 0$$

Using the *Zero Property Principle* we get

$2x = 0$ or $x - 3 = 0$

and $x = 0$ or $x = 3$

Since not both equations are linear, we must find the values for y using both original equations.

Using the linear equation $y = x - 3$:

- If $x = 0$, then $y = 0 - 3 = -3$

  Thus, one solution is $(0, -3)$.

- If ***x = 3***, then ***y = 3 - 3 = 0***

  and the second solution is ***(3, 0)***.

Using the circle $x^2 + y^2 = 9$:

- If ***x = 0***, then

  $0^2 + y^2 = 9$
  $y^2 = 9$
  $y = \pm 3$

  Please notice that ***3*** must be an **extraneous solution** because we did not find it when using the linear equation.

- If ***x = 3***, then

  $3^2 + y^2 = 9$
  $y^2 = 0$
  $y = 0$

Therefore, the solutions to the given system are ***(0, -3)*** and ***(3, 0)***.

Please note the following:

Given the circle $x^2 + y^2 = 9$, we know that its center is at ***(0, 0)*** and its radius is ***3***. Therefore, its x-intercepts must be at ***(-3, 0)*** and ***(3, 0)*** and its y-intercept at ***(0, -3)*** and ***(0, 3)***.

Given the solutions to the system of equations, we find that the line $y = x - 3$ must intercept the x- and y-axis at two of the same points as the circle $x^2 + y^2 = 9$.