DETAILED SOLUTIONS AND CONCEPTS - MATRICES AND DETERMINANTS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definition of a Matrix

A matrix (plural: matrices) is a rectangular array of numbers arranged in rows and columns and placed in brackets, Each number in the matrix is called an **element**.

Examples of Matrices

$$A = \begin{bmatrix} 7 & -3 & 0 \\ 5 & 2 & -11 \end{bmatrix}$$ a matrix with 2 rows and 3 columns. We call this a 2 x 3 matrix (2 by 3)

The elements are: $a_{11} = 7$, $a_{12} = -3$, $a_{13} = 0$, $a_{21} = 5$, $a_{22} = 2$, $a_{23} = -11$

NOTE: The element names are subscripted where the first number indicates the row and the second number the column.

$$B = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$$ a 2 x 2 matrix

$$C = \begin{bmatrix} 4 & -9 & 0 \\ 6 & -1 & -2 \\ 0 & 8 & 11 \end{bmatrix}$$ a 3 x 3 matrix

NOTE: When the number of rows equals the number of columns we call the matrices "square matrices."

Determinant of a Matrix

Associated with every square matrix is a real number called the determinant. Its calculation changes depending on the number of rows and columns. Here we will only show the calculation of the determinant of a 2 x 2 matrix.

Calculation of the Deteminant of a 2 x 2 Matrix:

The determinant of a 2 x 2 matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Augmented Matrix

A matrix derived from a linear system of equations, each in standard from, is called the augmented matrix of the system. An augmented matrix has a vertical bar separating the columns of the matrix into two groups. The coefficients of each variable in a linear system are placed to the left of the vertical line, and the constants are placed to the right.

Here is an example of an augmented matrix given a system of three linear equations in three variables (The graphic representation of a linear equation in three variables is a plane in 3-space.)

System

$$\begin{aligned} 3x + y + 2z &= 31 \\ x + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 0 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

Augmented matrices can be used to solve systems of linear equations. The process is called Gaussian Elimination, after the German mathematician Carl Friedrich Gauss (1777-1855). This process is programmed into computers and expensive calculators to allow us to find solutions to systems of 50 or more equations in seconds!!!

Doing the matrix process by hand is extremely time-consuming. You must be very careful not to make sign errors!

Before you begin, you have to know three **matrix row operations** that you **MUST** carry out on an augmented matrix to achieve the solution to the system.

Description of the Operation	Symbol	Example
Interchange two rows of the matrix.	$R_i \longleftrightarrow R_j$	$\left[\begin{array}{cc\|c} 4 & -3 & -15 \\ 1 & 2 & -1 \end{array}\right]$ $R_1 \longleftrightarrow R_2$ Interchange Rows 1 and 2. $\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 4 & -3 & -15 \end{array}\right]$

Description of the Operation	Symbol	Example
To replace an element with 0: Add to the entries of any row a multiple of the corresponding entries of another row.	$kR_i + R_j$	$\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 4 & -3 & -15 \end{array}\right]$ $-4R_1 + R_2$ Multiply Row 1 by (-4) and add it to Row 2. $\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ -4(1)+4 & -4(2)+(-3) & -4(-1)+(-15) \end{array}\right]$ $\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 0 & -11 & -11 \end{array}\right]$

Description of the Operation	Symbol	Example
To replace an element with 1: Multiply the entries in any row by the same nonzero real number ***k***.	kR_i	$\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 0 & -11 & -11 \end{array}\right]$ $-\frac{1}{11}R_2$ Multiply Row 2 by $-\frac{1}{11}$ $\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 0 & 1 & 1 \end{array}\right]$

Use matrix row operations to simplify the augmented matrix to one with ***1s*** down the diagonal from upper left to lower right, and ***0s*** below the ***1s***. We call this is the **solution matrix**.

Following is the solution matrix for a system of three equations in three variables.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right]$$

where $x = A$, $y = B$, and $z = C$ is the solution to the system. Graphically, the solution is the point (A, B, C) in 3-space in which the three planes intersect.

In the next matrix, the small numbers next to the ***1s*** and ***0s*** indicate in which order they have to be derived.

NOTE: Any other order may cause you to do a lot of extra work!

$$\left[\begin{array}{ccc|c} 1_1 & 0_9 & 0_8 & A \\ 0_2 & 1_4 & 0_7 & B \\ 0_3 & 0_5 & 1_6 & C \end{array}\right]$$

Problem 1:

Solve the following system of equations using Gaussian Elimination. Express your answer as coordinates in 3-space (x, y, z).

$$\begin{aligned} -2x - 4y - 2z &= -18 \\ -4x - y + 2z &= 10 \\ 4x + 3y + 2z &= 10 \end{aligned}$$

Change the three equations to augmented matrix form:

$$\left[\begin{array}{ccc|c} -2 & -4 & -2 & -18 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right]$$

Note: The graph of each equation is a plane in three-dimensional space.

$\left[\begin{array}{ccc\|c} -2 & -4 & -2 & -18 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right]$	We want the number 1 in the first position of Row 1. Replace each element in Row 1 with the following calculations $-\frac{1}{2}R_1$
$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ -4 & -1 & 2 & 10 \\ 4 & 3 & 2 & 10 \end{array}\right]$	We want the number 0 in the first position of Row 2. Replace each element in Row 2 with the following calculations $4R_1 + R_2$
$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 4 & 3 & 2 & 10 \end{array}\right]$	We want the number 0 in the first position of Row 3. Replace each element in Row 3 with the following calculations $-4R_1 + R_3$
$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ 0 & 7 & 6 & 46 \\ 0 & -5 & -2 & -26 \end{array}\right]$	We want the number 1 in the second position of Row 2. Replace each element in Row 2 with the following calculations $\frac{1}{7}R_2$

$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & -5 & -2 & -26 \end{array}\right]$	We want the number 0 in the second position of Row 3. Replace each element in Row 3 with the following calculations $5R_2 + R_3$
$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & \frac{16}{7} & \frac{48}{7} \end{array}\right]$	We want the number 1 in the third position of Row 3. Replace each element in Row 3 with the following calculations $\frac{7}{16} R_3$
$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ 0 & 1 & \frac{6}{7} & \frac{46}{7} \\ 0 & 0 & 1 & 3 \end{array}\right]$	We want the number 0 in the third position of Row 2. Replace each element in Row 2 with the following calculations $-\frac{6}{7} R_3 + R_2$
$\left[\begin{array}{ccc\|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$	We want the number 0 in the third position of Row 1. Replace each element in Row 1 with the following calculations $-1R_3 + R_1$
$\left[\begin{array}{ccc\|c} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$	We want the number 0 in the second position of Row 1. Replace each element in Row 1 with the following calculations $-2R_2 + R_1$
$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$	The Solution Matrix. From it we can pick the results for x, y, and z.

Solution of system of equations: $x = -2$ $y = 4$ $z = 3$	**Graphically, the three planes in 3D space intersection at one point:** $(-2,4,3)$

Problem 2:

Solve the following system of equations using Gaussian Elimination. Express your answer as coordinates in 3-space (x, y, z).

$$\begin{aligned} x - 3y + z &= 1 \\ 2x - y - 2z &= 2 \\ x + 2y - 3z &= -1 \end{aligned}$$

Change the three equations to augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 2 & -1 & -2 & 2 \\ 1 & 2 & -3 & -1 \end{array}\right]$$

Note: The graph of each equation is a plane in three-dimensional space.

$\left[\begin{array}{ccc\|c} 1 & -3 & 1 & 1 \\ 0 & 5 & -4 & 0 \\ 1 & 2 & -3 & -1 \end{array}\right]$	Adding -2 times the first equation to the second equation produces a new second equation.

$\left[\begin{array}{ccc\|c} 1 & -3 & 1 & 1 \\ 0 & 5 & -4 & 0 \\ 0 & 5 & -4 & -2 \end{array}\right]$	Adding -1 times the first equation to the third equation produces a new third equation.

$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 5 & -4 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$	Adding -1 times the second equation to the third equation produces a new third equation.

Because 0 = -2 is a false statement, this system has **NO solutions**. **Graphically, the three planes in 3D space could be parallel to each other, or two planes each could intersect.**

Problem 3:

Solve the following system of equations using Gaussian Elimination. Express your answer as coordinates in 3-space (x, y, z).

$$\begin{aligned} x + y - 3z &= -1 \\ y - z &= 0 \\ -x + 2y &= 1 \end{aligned}$$

Change the three equations to augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right]$$

Note: The graph of each equation is a plane in three-dimensional space.

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$	Adding the first equation to the third equation produces a new third equation.

$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$	Adding -3 times the second equation to the third equation produces a new third equation.

Since the element in row 2 column 3 is not 1, we know that there must be an **infinite number of solutions**. **Graphically, the three planes in 3D space intersect in one line.**

However, it is not customary to say simply that the solution is "infinite." Usually a specify solution form is used.

Note, that the last matrix is equal to

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which can be written as

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Note that this can also be expressed as

$$\begin{aligned} x + y - 3z &= -1 \\ y - \ z &= \ \ 0 \end{aligned}$$

STEP 1:

In the last equation, which is ***y - z = 0***, we will solve for ***y*** in terms of ***z*** to obtain ***y = z***. Back-substituting for ***y*** into the first equation we get

$$x + (z) - 3z = -1$$

$$x - 2z = -1$$

and solving for x, we get

$$x = 2z - 1$$

STEP 2:

It is an accepted practice to let ***z = a*** (***a*** is any real number) in cases of infinite many solutions. Thus, if

z = a then ***y = a*** and ***x = 2a - a***.

Thus, every ordered triple of the form ***(2a - 1, a, a)*** is a solution of the system, ***a*** is any real number.

Problem 4:

Find the determinant of $\begin{bmatrix} 5 & 6 \\ 7 & 3 \end{bmatrix}$.

$$\begin{vmatrix} 5 & 6 \\ 7 & 3 \end{vmatrix} = 5(3) - 7(6) = -27$$

Problem 5:

Find the determinant of $\begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$.

$$\begin{vmatrix} 2 & 4 \\ -3 & -5 \end{vmatrix} = 2(-5) - (-3)(4) = 2$$

Problem 6:

Find the determinant of $\begin{bmatrix} -2 & 0 \\ -6 & 3 \end{bmatrix}$.

$$\begin{vmatrix} -2 & 0 \\ -6 & 3 \end{vmatrix} = -2(3) - (-6)(0) = -6$$