DETAILED SOLUTIONS AND CONCEPTS - SOLVING LOGARITHMIC EQUATIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

The following are examples of the two types of logarithmic equations that we will be solving in this section.

$$\log_4(x + 3) = 2 \qquad \log(2x - 1) = \log(4x + 3) - \log x$$

The first equation contains a logarithmic expression and a constant. The second equation contains only logarithmic expressions. Each type has its own strategies for solving.

Strategy for Solving Equations containing Logarithmic and Non-Logarithmic Expressions

- Collect all logarithmic expressions on one side of the equation and all constants on the other side.
- Use the *Rules of Logarithms* to rewrite the logarithmic expressions as the logarithm of a single quantity with coefficient of 1.
- Rewrite the logarithmic equation as an equivalent exponential equation.

  Remember that if $\log_b M = a$, then $M = b^a$!!!

- Solve for the variable.
- Check each solution in the original equation, rejecting apparent solutions that produce any logarithm of a negative number or the logarithm of 0. Usually, a visual check suffices!!!

  NOTE: The logarithm of 0 is undefined, and the logarithm of a negative number is defined to be an imaginary number. In this course, we are only interested in solutions to logarithmic equations that produce real numbers for logarithmic expressions.

Strategy for Solving Equations containing only Logarithmic Expressions

- Use the *Rules of Logarithms* to rewrite each side of the equation as the logarithm of a single quantity. The coefficients of the logarithms MUST be 1.
- By definition, if $\log_b M = \log_b N$, then $M = N$. In other words, you can "discard" the word **log** or **ln (el en)** on either side, leaving you with an algebraic equation.
- Solve for the variable.
- Check each solution in the **original** equation, rejecting apparent solutions that produce the logarithm of a negative number or the logarithm of 0. Usually, a visual check suffices!!!

Problem 1:

Solve $\log_4(x+3) = 2$. Only find solutions that produce REAL numbers, except 0, in the original equation when substituting for x!

First, we change the logarithmic equation to its exponential form

$x + 3 = 4^2$ (be sure to note that the logarithm base 4 becomes the exponent base for 2!)

$x = 16 - 3 = 13$

Check:

Let's substitute 13 for x in the original equation to make sure that

a. we don't get the logarithm of 0

b. we don't get the logarithm of a negative number

c. the solution is correct

$$\log_4(13+3) \stackrel{?}{=} 2$$
$$\log_4 16 \stackrel{?}{=} 2$$
$$\frac{\log 16}{\log 4} \stackrel{?}{=} 2$$
$$2 = 2$$

We find that $x = 13$ is an acceptable solution!

NOTE: A visual check of the **original** equation usually suffices to convince us that $x = 13$ will neither produce a logarithm of a negative number nor a logarithm of 0.

Problem 2:

Solve $\log(x-1) - \log(x+1) = 1$. Only find solutions that produce REAL numbers, except 0, in the original equation when substituting for x!

Before we can change to exponential form, we must first write one logarithm

$$\log\frac{x-1}{x+1} = 1$$

Now we can change to exponential form

$$\frac{x-1}{x+1} = 10^1$$

Cross multiplying, we find

$$x - 1 = 10(x+1)$$

$$x - 1 = 10x + 10$$

$$-9x = 11$$

$$x = -\tfrac{11}{9}$$

Check:

Let's substitute $-\frac{11}{9}$ for x in the original equation to make sure that

a. we don't get the logarithm of 0

b. we don't get the logarithm of a negative number

c. the solution is correct

$$\log(-\tfrac{11}{9} - 1) - \log(-\tfrac{11}{9} + 1) \stackrel{?}{=} 1$$

$$\log(-\tfrac{20}{9}) - \log(-\tfrac{2}{9}) \stackrel{?}{=} 1$$

$x = -\frac{11}{9}$ produces two logarithms of negative numbers, which we did not learn how to evaluate. Therefore, we will say in this course that the solution is NOT acceptable.

This equation has NO solutions.

Problem 3:

Solve $\ln x + \ln(2x-1) = 2$. Only find solutions that produce REAL numbers, except ***0***, in the original equation when substituting for x! Round to 4 decimal places.

$$\ln[x(2x-1)] = 2$$

$$x(2x-1) = e^2$$

$$2x^2 - x = e^2$$

$2x^2 - x - e^2 = 0$. This is a quadratic equation that is not factorable, therefore, we have to use the quadratic formula with

$$a = 2 \qquad b = -1 \qquad c = -e^2$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-e^2)}}{2(2)}$$

$$x = \frac{1 \pm \sqrt{1+8e^2}}{4}$$

$$x = \frac{1 + \sqrt{1+8e^2}}{4} \approx 2.1883$$

$$x = \frac{1 - \sqrt{1+8e^2}}{4} \approx -1.6883$$

Here, a visual check of the **original** equation indicates that $x \approx -1.6883$ will produce the logarithms of a negative number.

Therefore, $x = \frac{1 + \sqrt{1+8e^2}}{4} \approx 2.1883$ **is the only acceptable solution.**

Problem 4:

Solve $\log(4x + 2x^2) = \log(3x^2)$. Only find solutions that produce REAL numbers, except ***0***, in the original equation when substituting for x!

This logarithmic equation contains only logarithmic expressions. Therefore, we can discard the word ***log*** on either side, leaving us with an algebraic equation

$4x + 2x^2 = 3x^2$, which is quadratic. Therefore,

$$4x - x^2 = 0$$
$$x(4 - x) = 0$$

and $x = 0$ and $x = 4$.

Check:

Let's substitute 0 for x in the original equation.

$$\log[4(0) + 2(0)^2] \stackrel{?}{=} \log[3(0)^2]$$
$$\log 0 = \log 0$$

$\log 0$ is undefined whether the right side equals the left side or not. Therefore, $x = 0$ is NOT an acceptable solution.

Next, let's substitute 4 for x in the original equation.

$$\log[4(4) + 2(4)^2] \stackrel{?}{=} \log[3(4)^2]$$
$$\log 48 = \log 48$$

The right side equals the left side and $\log 48$ is defined and real. We find that $x = 4$ is an acceptable solution.

Therefore, $x = 4$ is the only acceptable solution.

Problem 5:

Solve $\log(2x - 1) = \log(4x + 3) - \log x$. Only find solutions that produce REAL numbers, except 0, in the original equation when substituting for x!

This logarithmic equation contains only logarithmic expressions. Therefore, we first have to rewrite the right side of the equation as the logarithm of a single quantity.

$$\log(2x - 1) = \log\frac{4x + 3}{x}$$

Now we can discard the word *log* on either side, leaving us with an algebraic equation

$$2x - 1 = \frac{4x + 3}{x}$$

Cross multiplying results in

$$x(2x - 1) = 4x + 3$$

$$2x^2 - x = 4x + 3$$

$$2x^2 - 5x - 3 = 0$$

This is factorable as follows

$$(2x + 1)(x - 3) = 0$$

$2x + 1 = 0$ or $x - 3 = 0$

$x = -\frac{1}{2}$ or $x = 3$

A visual check of the **original** equation indicates that $x = -\frac{1}{2}$ will produce one logarithm of a negative number.

Therefore, $x = 3$ is the only acceptable solution.

Problem 6:

Solve $\log(x + 4) - \log x = \log(x + 2)$. Only find solutions that produce REAL numbers, except 0, in the original equation when substituting for x! Round to 4 decimal places.

$$\log \frac{x + 4}{x} = \log(x + 2)$$

$$\frac{x + 4}{x} = x + 2$$

Using cross multiplication, we get

$$x + 4 = x(x + 2)$$

$$x + 4 = x^2 + 2x$$

$$0 = x^2 + x - 4$$

This is a quadratic equation that is not factorable, therefore, we have to use the quadratic formula

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{17}}{2}$$

That is, $x = \frac{-1+\sqrt{17}}{2} \approx 1.5616$

and $x = \frac{-1-\sqrt{17}}{2} \approx -2.5616$

A visual check of the **original** equation indicates that $x \approx -2.5616$ will produce one logarithm of a negative number.

Therefore, $x = \frac{-1+\sqrt{17}}{2} \approx 1.5616$ **is the only acceptable solution.**

Problem 7:

Solve for **x**: $2\ln x = \ln(2-x) + \ln(4-x)$. Only find solutions that produce REAL numbers, except ***0***, in the original equation when substituting for ***x*** !

We first have to rewrite the right side of the equation as the logarithm of a single quantity. We also have to rewrite the left side as a logarithm with ***coefficient 1***,

$$\ln x^2 = \ln[(2-x)(4-x)]$$

$$x^2 = (2-x)(4-x)$$

$$x^2 = 8 - 6x + x^2$$

$$6x = 8$$

$$x = \tfrac{8}{6} = \tfrac{4}{3}$$

A visual check of the **original** equation indicates that $x = \frac{4}{3}$ will neither produce a logarithm of a negative number nor a logarithm of ***0***.

Therefore, $x = \frac{4}{3}$ **is an acceptable solution.**

Problem 8:

Solve $\log(x-3) = \log(-x)$. Only find solutions that produce REAL numbers, except ***0***, in the original equation when substituting for ***x*** !

Since both sides of the equation are already single logarithms, we can write

$$x - 3 = -x$$

$$2x = 3$$

and $x = \frac{3}{2}$

A visual check of the **original** equation indicates that $x = \frac{3}{2}$ will produce the logarithms of a negative number.

Therefore, this equation has NO solutions.

Problem 9:

Solve $\log(3 - \frac{1}{2}x) = \log(-x)$. Only find solutions that produce REAL numbers, except 0, in the original equation when substituting for x!

Since both sides of the equation are already single logarithms, we can write

$$3 - \frac{1}{2}x = -x$$
$$3 = -\frac{1}{2}x$$

and $x = -6$

A visual check of the **original** equation indicates that $x = -6$ will neither produce a logarithm of a negative number nor a logarithm of 0.

Therefore, $x = -6$ is an acceptable solution.

Problem 10:

Solve $\log x^2 = \log(-5x - 6)$. Only find solutions that produce REAL numbers, except 0, in the original equation when substituting for x!

Since both sides of the equation are already single logarithms, we can write

$$x^2 = -5x - 6$$
$$x^2 + 5x + 6 = 0$$

$$(x + 3)(x + 2) = 0$$

Then $x + 3 = 0$ and $x + 2 = 0$, which produces $x = -3$ and $x = -2$

A visual check of the **original** equation indicates that neither solution produces a logarithm of a negative number nor a logarithm of 0.

Therefore, $x = -3$ and $x = -2$ are acceptable solutions.

Problem 11:

Solve $2\log x = \log(-5x-6)$. Only find solutions that produce REAL numbers, except ***0***, in the original equation when substituting for x!

Before we can solve this equation we MUST use the *Power Rule* to change the coefficient of the logarithmic term on the left to ***1***.

That is, $\log x^2 = \log(-5x-6)$. **Please note that this is now the equation from Problem 10!**

Continuing just as in Problem 10, we find the solutions to be $x = -3$ and $x = -2$.

However, in this case, a visual check of the **original** equation indicates that both solutions produce a logarithm of a negative number on the left.

Therefore, this equation has NO solutions.

Problem 12:

A medical technologist creates a reagent with a ***pH*** of 7.48. Find the concentration of hydrogen ions $[H^+]$ in the reagent using the formula $pH = -\log[H^+]$. Express your answer in *Scientific Notation* rounded to two decimal places.

$$7.48 = -\log[H^+]$$

multiply both sides by ***-1***

$$-7.48 = \log[H^+]$$

change the logarithmic form to exponential form

$$H^+ = 10^{-7.48} \approx 3.31 \times 10^{-8}$$

The hydrogen ion concentration is approximately 3.31×10^{-8}.