DETAILED SOLUTIONS AND CONCEPTS - INVERSE ALGEBRAIC FUNCTIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

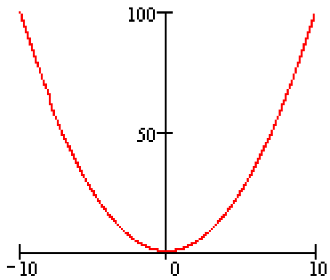
If a function is one-to-one, its inverse function may be obtained by interchanging x and y of the given function. In this reversal, the domain of the given function becomes the range of the inverse function, and the range of the given function becomes the domain of the inverse function.

If every horizontal line intersects the graph of a function in at most one point, then the function is one-to-one. That is, for every y-value there exists at most one x-value.

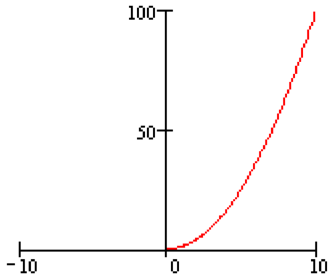
For example, quadratic and absolute value functions are NOT one-to-one functions. Linear and cubic functions, on the other hand, are one-to-one functions.

If a function is NOT *one-to-one*, we can still find an inverse, as long as we restrict the domain of the function to a *one-to-one* portion.

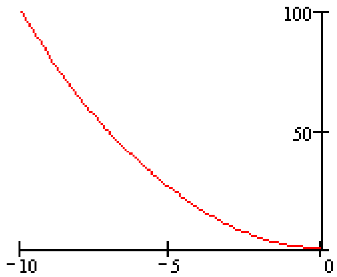
For example, the function $f(x) = x^2$ is not one-to-one since any horizontal line intersects with the graph more than once



However, if we restrict the domain of $f(x) = x^2, x \geq 0$, then we have a function that is *one-to-one*.



We can also restrict the domain another way. How about $f(x) = x^2, x \le 0$? We will also get a *one-to-one* function.



Cancellation Properties of Inverse Functions

If f is a *one-to-one* function, the inverse function f^{-1} satisfies the following cancellation properties:

$f^{-1}[f(x)] = x$ for every x in the domain of f

and

$f[f^{-1}(x)] = x$ for every x in the domain of f^{-1}

NOTE: The notation $f^{-1}(x)$ is referred to as ***f inverse of x***. It does NOT mean $\frac{1}{f(x)}$!!!!!

Strategy for Finding the Inverse of a One-To-One Function

- If necessary, replace the function notation by y in the given equation.
- Interchange x and y.
- Solve for y in terms of x.
- Replace y by function notation. You can use $f^{-1}(x)$ if you want, or any other function notation.

Graphically, a function and its inverse are symmetric with respect to the line $y = x$.

Proof:

Suppose that ***(a, b)*** is a point on the graph of a one-to-one function ***f***.

Then ***b = f(a)***. This means that $a = f^{-1}(b)$.

So ***(b, a)*** is a point on the function $f^{-1}(x)$.

The equation of the line segment from ***(a, b)*** to ***(b, a)*** is ***y = - x + (a + b)***.

It has slope ***m = -1*** and is, therefore, perpendicular to the line ***y = x***.

The midpoint of the line segment from ***(a, b)*** to ***(b, a)*** is at $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$.

Only for the line ***y = x*** the y-coordinate equals the x-coordinate for any given value of ***x***.

Problem 1:

Given the *one-to-one* function $y = \sqrt{x - 2}$ with domain $x \geq 2$ and range $y \geq 0$, find $f^{-1}(x)$ with its domain and range.

Let's exchange ***x*** and ***y***.

$x = \sqrt{y - 2}$ This is the inverse function.

To find $f^{-1}(x)$, we must solve for ***y*** in terms of ***x***.

When we raise both sides of the equation to the second power, we get

$x^2 = y - 2$

and $y = x^2 + 2$.

Thus, the inverse of $y = \sqrt{x - 2}$ is $f^{-1}(x) = x^2 + 2$ with a domain of ***{ x /*** $x \geq 0$ ***}*** and a range of ***{ y /*** $y \geq 2$ ***}***.

Remember that the range of a function becomes the domain of its inverse function. Likewise, the domain of a function becomes the range of its inverse function.

Problem 2:

Given the *one-to-one* function $f(x) = \sqrt[3]{x-5}$ with domain and range consisting of ***All Real Numbers***, find $f^{-1}(x)$ with its domain and range.

If $y = \sqrt[3]{x-5}$, then

$x = \sqrt[3]{y-5}$ This is the inverse function.

To find $f^{-1}(x)$, we must solve for y in terms of x.

When we raise both sides of the equation to the third power, we get

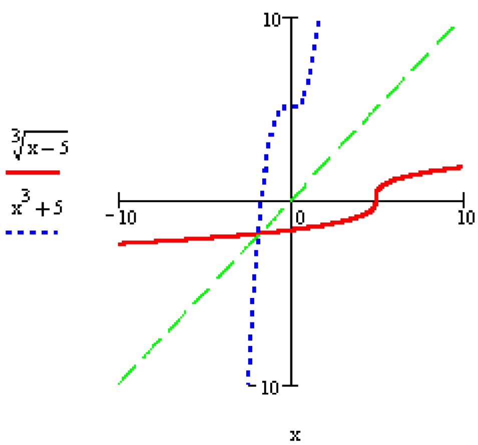
$$x^3 = y - 5$$

$$y = x^3 + 5.$$

Thus, the inverse of the given function is $f^{-1}(x) = x^3 + 5$ with a domain and range consisting of ***All Real Numbers***.

Let's look at the graph of $f^{-1}(x) = x^3 + 5$ and $f(x) = \sqrt[3]{x-5}$.

As you can see the two graphs are symmetric with respect to the line $y = x$. The "doted" (blue) graph is the graph of f and the solid (red) graph is the graph of f^{-1}.



Problem 3:

Given the *one-to-one* function $f(x) = 3x - 2$ with the domain and range consisting of ***All Real Numbers***, find $f^{-1}(x)$ with its domain and range.

If $y = 3x - 2$, then

$x = 3y - 2$ This is the inverse function.

To find $f^{-1}(x)$, we must solve for y in terms of x

$x + 2 = 3y$

$y = \dfrac{x+2}{3}$.

Thus, the inverse of the given function is $f^{-1}(x) = \dfrac{x+2}{3} = \dfrac{1}{3}x + \dfrac{2}{3}$ with a domain and range consisting of ***All Real Numbers***.

Problem 4:

Given the *one-to-one* function $f(x) = \dfrac{2x-3}{x+4}$ with domain $\{x \mid x \neq -4\}$ and range $\{y \mid y \neq 2\}$, find $f^{-1}(x)$ with its domain and range.

If $y = \dfrac{2x-3}{x+4}$ then

$x = \dfrac{2y-3}{y+4}$ This is the inverse function.

To find $f^{-1}(x)$, we must solve for y in terms of x.

Multiplying both sides by the common denominator, we find

$$x(y+4) = 2y - 3$$
$$xy + 4x = 2y - 3$$

Bringing all terms containing y to one side and the remaining terms to the other side, we get

$$xy - 2y = -4x - 3$$

Factoring the y-variable out of the left side

$$y(x-2) = -4x - 3$$

and, finally, we divide both sides by $(x - 2)$ to get

$$y = \frac{-4x-3}{x-2}$$

Thus, the inverse of the given function is $f^{-1}(x) = \frac{-4x-3}{x-2}$ with domain $\{x \mid x \neq 2\}$ and range $\{y \mid y \neq -4\}$.

NOTE: The inverse can be written in several different, but equal ways.

That is, we could factor out a negative 1 in the numerator and place it in front of the fraction to get

$$y = -\frac{4x+3}{x-2}.$$

Or we could factor out a negative 1 in the numerator and send it into the denominator to get $y = \frac{4x+3}{2-x}$.

Problem 5:

Given the *one-to-one* function $g(x) = \frac{4x}{5x-9}$ with domain $\{x \mid x \neq \frac{9}{5}\}$ and range $\{y \mid y \neq \frac{4}{5}\}$, find $g^{-1}(x)$ with its domain and range.

If $y = \frac{4x}{5x-9}$ then

$x = \frac{4y}{5y-9}$ This is the inverse function.

To find $f^{-1}(x)$, we must solve for y in terms of x.

Multiplying both sides by the common denominator, we find

$$x(5y-9) = 4y$$
$$5xy - 9x = 4y$$

Bringing all terms containing y to one side and the remaining terms to the other side, we get

$$5xy - 4y = 9x$$

Factoring the y-variable out of the left side

$$y(5x - 4) = 9x$$

and, finally, we divide both sides by $(5x - 4)$ to get

$$y = \frac{9x}{5x - 4}$$

Thus, the inverse of the given function is $g^{-1}(x) = \frac{9x}{5x - 4}$ with domain $\{x \mid x \neq \frac{4}{5}\}$ and range $\{y \mid y \neq \frac{9}{5}\}$.

Problem 6:

Given the *one-to-one* function $g(x) = \frac{1}{x}$ with domain $\{x \mid x \neq 0\}$ and range $\{y \mid y \neq 0\}$, find $g^{-1}(x)$ with its domain and range.

If $y = \frac{1}{x}$ then

$x = \frac{1}{y}$ This is the inverse function.

To find $f^{-1}(x)$, we must solve for y in terms of x

Multiplying both sides by the common denominator, we find

$$xy = 1$$

and dividing both sides by x, we get $y = \frac{1}{x}$.

Thus, the inverse of the given function is $g^{-1}(x) = \frac{1}{x}$ with domain $\{x \mid x \neq 0\}$ and range $\{y \mid y \neq 0\}$.

Please note that a function can be its own inverse!

Problem 7:

Show that the functions ***f*** and ***g*** are inverses of each other. Note that the notation f^{-1} does not necessarily have to be used to indicate an inverse.

$f(x) = \sqrt{x-9} + 7, \quad x \geq 9$ and $g(x) = x^2 - 14x + 58, \quad x \geq 7$

We must show that ***f [g(x)] = x*** and ***g[f(x)] = x*** to prove that the two functions are inverses of each other.

Show that ***f [g(x)] = x***

We know that $f[g(x)] = f(x^2 - 14x + 58)$.

This means that we must replace the variable ***x*** in the function $f(x) = \sqrt{x-9} + 7$ with $x^2 - 14x + 58$.

$$\begin{aligned} f[g(x)] &= \sqrt{(x^2 - 14x + 58) - 9} + 7 \\ &= \sqrt{x^2 - 14x + 49} + 7 \\ &= \sqrt{(x-7)^2} + 7 \\ &= (x-7) + 7 \\ &= x \end{aligned}$$

Show that ***g [f(x)] = x***

We know that $g[f(x)] = g(\sqrt{x-9} + 7)$.

This means that we must replace the variable ***x*** in the function $g(x) = x^2 - 14x + 58$ with $\sqrt{x-9} + 7$.

$$\begin{aligned} g[f(x)] &= \left(\sqrt{x-9} + 7\right)^2 - 14\left(\sqrt{x-9} + 7\right) + 58 \\ &= (x-9) + 14\sqrt{x-9} + 49 - 14\sqrt{x-9} - 98 + 58 \\ &= x - 9 + 49 - 98 + 58 \\ &= x \end{aligned}$$

Since both cancellation properties are satisfied, the two functions are said to be inverses of each other.

Problem 8:

Show that the functions ***f*** and ***g*** are **NOT** inverses of each other.

$$f(x) = \frac{3}{x} \text{ and } g(x) = \frac{x}{3}$$

We must show that ***f [g(x)] = x*** and ***g[f(x)] = x***. If this is not possible, we can conclude that the two functions are NOT inverses of each other.

Show that ***f [g(x)] = x***

$$f[g(x)] = \frac{3}{\frac{x}{3}} = \frac{3}{1} \cdot \frac{3}{x} = \frac{9}{x}$$

Since $f[g(x)] \neq x$, we can stop and immediately conclude that the two functions are **NOT** inverses of each other.

Problem 9:

Given the *one-to-one* function $f(x) = 5 - x$ with the domain and range consisting of ***All Real Numbers***, find $f^{-1}(0)$.

Let's first find $f^{-1}(x)$.

If $y = 5 - x$, then

$x = 5 - y$ This is the inverse function.

$$x - 5 = -y$$

$$y = -x + 5$$

and $f^{-1}(x) = -x + 5$

Now we can find $f^{-1}(0)$.

$$f^{-1}(0) = 0 + 5 = 5$$