DETAILED SOLUTIONS AND CONCEPTS - INTRODUCTION TO FUNCTIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Mathematics once revolved around the concept of an equation, but eventually the more sophisticated notion of "function" evolved.

Definition of Function:

Given an equation in two unknowns, say x and y, a function is a rule that assigns to each element x EXACTLY ONE element y.

Independent and Dependent Variables:

In mathematics, unless otherwise stated, x is usually called the **independent variable** and y the **dependent variable**. For graphing purposes, the independent variable is assigned to the horizontal axis in a *Cartesian Coordinate System*, and the dependent variable to the vertical axis.

We usually choose real numbers for the independent variable. The numeric values of the dependent variable then depend on our pick of numbers for the independent variable.

Domain:

The domain of a function is the set of all **real number replacements** for the independent variable over which the function is defined. Therefore, the domain MUST exclude all values that make the dependent variable imaginary and/or undefined.

At this point, we are discussing algebraic functions. For them, the only way we would have to consider the possibility of imaginary y-values is when the function contains radicals with even index, for Problem, $\sqrt[4]{\ }, \sqrt[6]{\ }, \ldots, \sqrt[100]{\ }$, etc.

Therefore, every time you encounter variables in the radicand of a radical with even index, you simply find the set of solutions that make the radicand greater than or equal to 0.

Furthermore, we only have to worry about undefined y-values when the function contains variables in the denominator. Remember, $\frac{1}{0}$ = undefined.

Therefore, every time you encounter variables in the denominator, you simply set the denominator equal to *0* and solve for the variable. The resulting numbers MUST be excluded from the domain.

Range:

The set of numbers for the dependent variable resulting from our pick for the independent variable is the **range** of the function.

Function Notation:

If an equation in two variables, say x and y, is a function, we can replace the y-variable with function notation.

Most common is $f(x)$, but you can use other letters of the alphabet. For Problem, you could also write $g(x)$ or $P(x)$. The notation $f(x)$ is referred to as ***f of x***. It describes the y-value of the function named f at the independent variable x.

NOTE: The independent variable does not need to be named x. For Problem, the function notations $g(r)$, $S(m)$, $F(t)$ indicate that the independent variables are r, m, and t respectively.

Some Problems of Equations that are Functions

Linear Functions:

$f(x) = mx + b$, where $m \neq 0$

The graph of the linear function is a line with positive or negative slope m.

The domain of the linear function consists of *All Real Numbers*.

Constant Functions:

$g(x) = b$, where b is a constant.

A constant function is a special linear function in which $m = 0$. That is, the graph of a constant function is a horizontal line.

The domain of the constant function consists of *All Real Numbers*.

Semicircular Functions:

Given the standard equation of the circle $(x-h)^2 + (y-k)^2 = r^2$, where r is the radius and (h, k) is the center of the circle, we will solve for y to find

$y = \sqrt{r^2 - (x-h)^2} + k$, which is the upper half of the circle and considered a function and

$y = -\sqrt{r^2 - (x-h)^2} + k$, which is the lower half of the circle and also considered a function

The domain of the semicircular function consists of numbers in the interval $[h-r, h+r]$.

Special Case of the Semicircular Function:

A circle with the standard equation of $x^2 + y^2 = r^2$, where ***r*** is the radius, has its center at the origin ***(0, 0)***. Its domain is ***[-r, r]***.

Some Problems of Equations that are NOT Functions

Vertical Lines:

x = a. Vertical lines are not functions because for each x-value there are infinitely many y-values.

Circles:

Circles are also not functions because for every x-value there are two y-values.

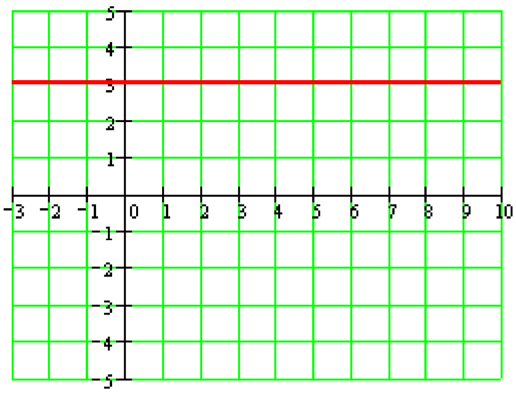
Identifying Functions by Examining their Graphs

The Vertical Line Test

A set of points in the *Cartesian Coordinate System* is the graph of a function if and only if every vertical line intersects with the graph in at most one point.

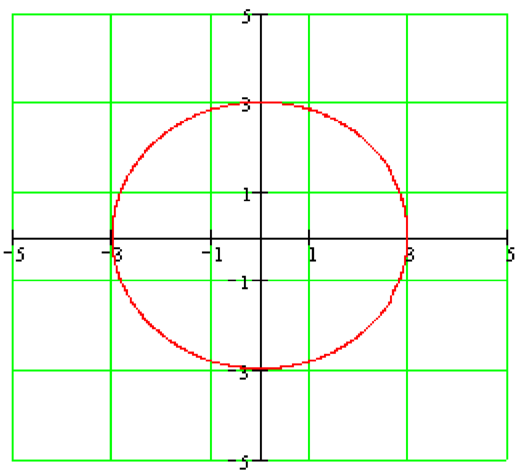
Look at the intersection of the vertical gridlines with the following graphs. Which ones depict functions and which ones do not?

Horizontal Line $y = 3$



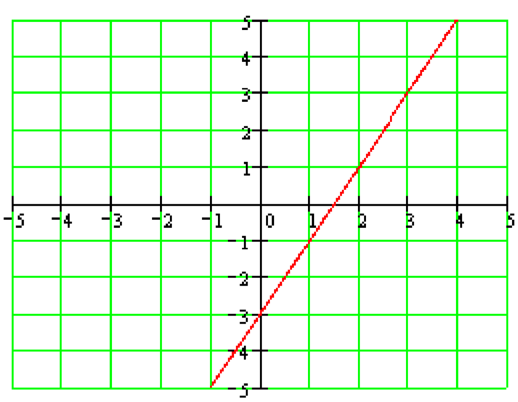
the graph is a function

Circle $x^2 + y^2 = 9$



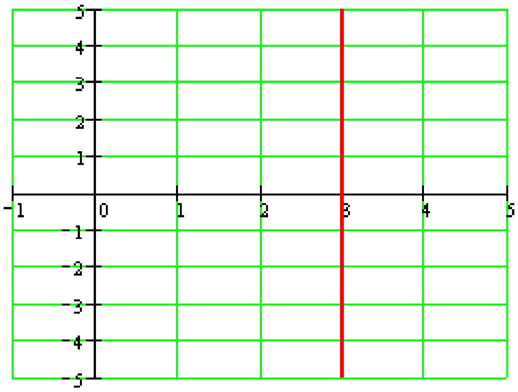
the graph is **NOT** a function

Oblique Line $y = 2x - 3$



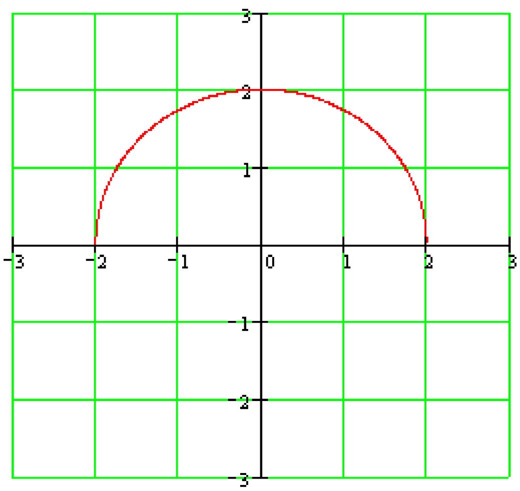
the graph is a function

Vertical Line $x = 3$



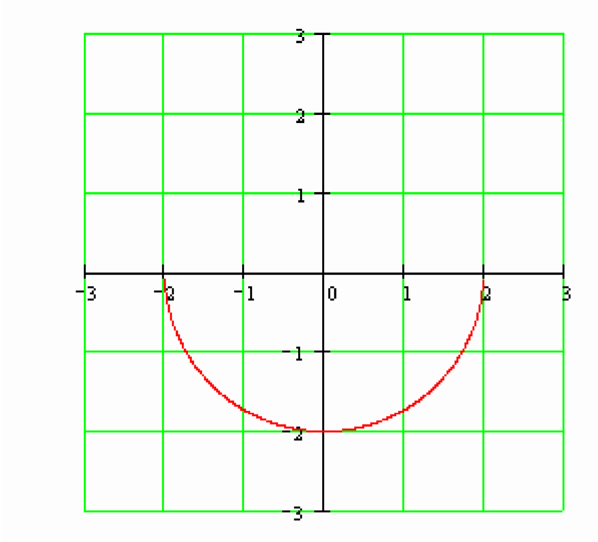
the graph is **NOT** a function

Semicircle $y = \sqrt{4 - x^2}$ - Upper half of the circle $x^2 + y^2 = 4$



the graph is a function

Semicircle $y = -\sqrt{4 - x^2}$ - Lower half of the circle $x^2 + y^2 = 4$



the graph is a function

Problem 1:

Given $f(x) = 2x - 3$, find ***f(5)***.

f(5) is the y-value that corresponds to ***x = 5***

In our Problem, $f(5) = 2(5) - 3 = 7$, that is, $f(5) = 7$

which means that when ***x = 5*** then ***y = 7***.

Problem 2:

Given $G(m) = 3 - m^2$, find $G(-1)$.

$G(-1) = 3 - (-1)^2 = 3 - 1 = 2$, that is, $G(-1) = 2$

which means that when ***m = -1*** then the dependent variable, say ***n = 2***.

Problem 3:

Given ***g(t) = 5***, find ***g(0)***, ***g(-10)***, and ***g(20)***.

g(0) = 5

g(-10) = 5

g(20) = 5

The function ***g*** is a constant function. Its graph is a horizontal line. For every x-value there is exactly one y-value, but all y-values are the same!

Problem 4:

Given $f(x) = x^a$, find $f(x^a)$.

Here we are asked to replace the variable ***x*** in the original equation with the exponential expression x^a.

That is,

$$f(x^a) = (x^a)^a$$
$$= x^{a^2}$$

Problem 5:

Given the function $f(x) = 3x^2 + 4x - 5$, evaluate ***f(a), f(-a), f(a + h), f(2x - 3), f(x + h)***.

a. $f(a)$

$f(a) = 3a^2 + 4a - 5$. We replaced all x-variables with ***a***, which is the final form.

b. $f(-a)$

$f(-a) = 3(-a)^2 + 4(-a) - 5$. We replaced all x-variables with ***-a***.

Multiplying the terms, we get a final form

$$f(-a) = 3a^2 - 4a - 5$$

c. $f(a + h)$

$f(a + h) = 3(a + h)^2 + 4(a + h) - 5$. We replaced all x-variables with ***(a + h)***

Multiplying the terms, we get

$$f(a + h) = 3(a^2 + 2ah + h^2) + 4(a + h) - 5$$

and

$$f(a + h) = 3a^2 + 6ah + 3h^2 + 4a + 4h - 5$$

d. $f(2x-3)$

$$\begin{aligned} f(2x-3) &= 3(2x-3)^2 + 4(2x-3) - 5 \\ &= 3(4x^2 - 12x + 9) + 8x - 12 - 5 \\ &= 12x^2 - 36x + 27 + 8x - 17 \\ &= 12x^2 - 28x + 10 \end{aligned}$$

Please note that it is acceptable practice to write *f(x + h)* only once and in subsequent steps just line up the equal signs. However, if you want you can also write the following:

$$f(2x-3) = 3(2x-3)^2 + 4(2x-3) - 5$$
$$f(2x-3) = 3(4x^2 - 12x + 9) + 8x - 12 - 5$$
$$f(2x-3) = 12x^2 - 36x + 27 + 8x - 17$$
$$f(2x-3) = 12x^2 - 28x + 10$$

Please note that in this case it is NOT acceptable practice to run the steps horizontally because each step is considered too long and too complex! For comparison purposes see Problems 1, 2, 4, and 6 - 10!

e. $f(x+h)$

$$\begin{aligned} f(x+h) &= 3(x+h)^2 + 4(x+h) - 5 \\ &= 3(x^2 + 2xh + h^2) + 4x + 4h - 5 \\ &= 3x^2 + 6xh + 3h^2 + 4x + 4h - 5 \end{aligned}$$

Problem 6:

Given $f(x) = \sqrt{x-5} + 4$, find $f(17)$.

$f(17) = \sqrt{17-5} + 4 = \sqrt{12} + 4 = 2\sqrt{3} + 4$, that is, $f(17) = 2\sqrt{3} + 4$

Problem 7:

Given $g(x) = 3\,|2x-3| - 6$, find $g(1)$.

$g(1) = 3\,|2(1)-3| - 6 = 3\,|-1| - 6 = 3(1) - 6 = -3$, that is, $g(1) = -3$

Problem 8:

Given $h(x) = -(2x+4)^{1/3} - 3$, find ***h(2)***.

$h(2) = -[2(2)+4]^{1/3} - 3 = -\sqrt[3]{8} - 3 = -2 - 3 = -5$, that is, ***h(2) = -5***

Problem 9:

Given $p(x) = -x^2 - 2x + 9$, find ***p(3)***.

$p(3) = -(3)^2 - 2(3) + 9 = -9 - 6 + 9 = -6$, that is, ***p(3) = -6***

Problem 10:

Given $k(x) = (x+8)^{2/3} + 2$, find ***k(19)***.

$k(19) = (19+8)^{2/3} + 2 = (\sqrt[3]{27})^2 + 2 = 9 + 2 = 11$, that is, ***k(19)= 11***

Problem 11:

Given $f(x) = 3x^3 - 5x^2 + x - 3$, find $f(-x)$.

$$f(-x) = 3(-x)^3 - 5(-x)^2 + (-x) - 3$$
$$= 3(-x^3) - 5(x^2) - x - 3$$
$$= -3x^3 - 5x^2 - x - 3$$

Problem 12:

Find the domain of the function $g(x) = \dfrac{3}{x^2 + 2x - 15} + x$ in *Set-Builder Notation* over which the function is defined.

Since we are required to let ***g(x) = y*** equal a real number, we have to find all numbers that make the denominator NOT equal to ***0*** because they would cause the y-value to be undefined.

Please note that we will only use the denominator for the domain calculation since IT is the only part of the function that could create undefined y-values.

$$x^2 + 2x - 15 = 0$$

$$(x+5)(x-3) = 0$$

$x + 5 = 0$ or $x - 3 = 0$

$x = -5$ or $x = 3$, which are the numbers that make the denominator equal to 0. These two numbers have to be excluded from the domain.

Therefore, the domain of the function is $\{x \mid x \neq -5, x \neq 3\}$.

Problem 13:

Find the domain of the function $f(x) = \sqrt{3x+12} - 1$ in *Interval Notation* over which the function is defined.

Any time you are presented with radicals of even index you have to worry about imaginary numbers. For Problem, the square root (index 2) of a negative number results in an imaginary number.

Since we are required to keep the y-value a real number, we have to find all numbers so that the radicand is greater than or equal to 0, that is, positive.

Please note that we will only use the radical for the domain calculation since IT is the only part of the function that could create imaginary y-values.

$3x + 12 \geq 0$

$3x \geq -12$

$x \geq -4$ This means that all number replacements for x that are greater than or equal to -4 will cause the radicand to be either positive or 0, which, in turn, gives us a real y-value.

Therefore, the domain of the function is $[-4, \infty)$.

Problem 14:

Find the domain of the function $g(x) = \sqrt{4 - \frac{1}{2}x}$ in *Interval Notation* over which the function is defined.

Again, we have to worry about x-values that will make the y-value imaginary. Therefore, we must find all of the x-values that turn the radicand into either a positive number or the number 0. We will write this as follows:

$4 - \frac{1}{2}x \geq 0$ In English this inequality means "find the numbers that make the radicand greater than or equal to 0, that is, positive.

Remember that we solve inequalities very much like equalities except when it comes to multiplying or dividing both sides by a negative number. As we will see very soon.

So let's first subtract 4 from both sides to get

$-\frac{1}{2}x \geq -4$

Now, we need to multiply both sides by ***-2***. Since we are dealing with an inequality, there now occurs a sign change. That is, the inequality sign changes direction as follows:

$x \leq -4(-2)$ Please check out the lecture notes entitled "Solving Inequalities" in the Prerequisites.

We find $x \leq 8$. This means that all number replacements for x that are less than or equal to 8 will cause the radicand to be either positive or 0, which, in turn, gives us a real y-value.

Therefore, the domain of the function is $(-\infty, 8]$.

Problem 15:

Find the domain of the function $k(x) = 2x - 3$ in *Interval Notation* over which the function is defined.

Since there is no denominator and no radicals with even index the domain is $(-\infty, \infty)$.

Problem 16:

Find the domain of the function $f(t) = t^2 - 3t + 1$ in *Interval Notation* over which the function is defined.

Since there is no denominator and no radicals with even index the domain is $(-\infty, \infty)$.

Problem 17:

Find the domain of the function $g(t) = 5$ in *Interval Notation* over which the function is defined.

Since there is no denominator and no radicals with even index the domain is $(-\infty, \infty)$.

Problem 18:

Find the domain of the function $S(p) = \frac{4 - p}{p^2 + 5}$ in *Interval Notation* over which the function is defined.

Since we are required to let ***S(p)*** equal a real number, we have to find all numbers that make the denominator NOT equal to 0 because they would cause the y-value to be undefined.

$$p^2 + 5 = 0$$
$$p^2 = -5$$
$$p = \pm\sqrt{-5} = \pm i\sqrt{5}$$

Since imaginary numbers are never in the domain of any function we conclude that the domain is $(-\infty, \infty)$.

Problem 19:

Find the domain of the function $y = x^2\sqrt{9 - x^2}$ in *Interval Notation* over which the function is defined.

Please note that we will only use the radical for the domain calculation since IT is the only part of the function that could create imaginary y-values.

Noticing that the graph of the radical by itself is a semicircle with center at the origin and radius 3, we know that the domain must be $[-3,3]$.

Problem 20:

Find the domain of the function $f(x) = \sqrt[3]{x^2 - 4}$ in *Interval Notation* over which the function is defined.

Since there is no denominator and no radicals with **even** index the domain is $(-\infty, \infty)$.

Problem 21:

Find the domain of $h(x) = -(2x+4)^{1/3} - 3$ in *Interval Notation* over which the function is defined.

Since we can write this function as $h(x) = -\sqrt[3]{2x+4} - 3$, which contains a radical with odd index and no denominator, the domain is $(-\infty, \infty)$.

Problem 22:

Find the domain of $k(x) = (x+8)^{2/3} + 2$ in *Interval Notation* over which the function is defined.

Since we can write this function as $k(x) = \sqrt[3]{(x+8)^2} + 2$ which contains a radical with odd index and no denominator, the domain is $(-\infty, \infty)$.

Problem 23:

Find the domain of the absolute value function $g(x) = 3\,|\,2x - 3\,| - 6$ in *Interval Notation* over which the function is defined.

Since there is no denominator and no radicals with even index the domain is $(-\infty, \infty)$.

Problem 24 (Linear Function):

The height h of a female is related to the length of her femur f (bone from the hip socket to the knee) by the mathematical model $f = 0.432h - 10.44$. Both h and f are measured in inches. Part of a female skeleton is found in which the femur is 18 inches long. How tall must this woman have been? Express your answer in feet and inches rounded to whole numbers.

$$18 = 0.432h - 10.44$$

$$h = \frac{18 + 10.44}{0.432} \approx 65.83$$

The dead woman must have been approximately *5 ft 6 in* tall.