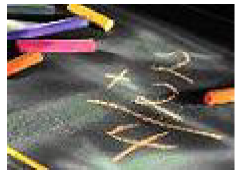

DETAILED SOLUTIONS AND CONCEPTS - EXPONENTIAL FUNCTIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definition of the Exponential Function

The exponential function f with base b is defined by

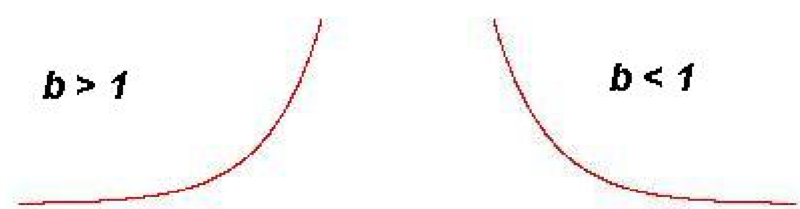
$y = b^x$, where b is a positive number other than 1

The domain of the exponential functions consists of all real numbers.

Characteristics of Graphs of Exponential Functions

Most exponential functions $f(x) = b^x$ and their transformations are best graphed with a graphing utility because the y-values get extremely large/small very quickly and are difficult to show in a hand-drawn *Cartesian Coordinate System*.

The graph of $f(x) = b^x$ has the following shapes.



- The graph consists of a SMOOTH curve with a rounded turn.
- Exponential functions and their transformations have *horizontal asymptotes*.
- The equation of the *horizontal asymptote* of the graph of $f(x) = b^x$ is $y = 0$, which is the x-axis.
- ONLY vertical shifts of the graph of $f(x) = b^x$ change the equation of the *horizontal asymptote*.
- The graph is never parallel to the y-axis, but moves away from it at a steady pace.
- The graph is never parallel to the horizontal asymptote, but moves toward it at a steady pace.

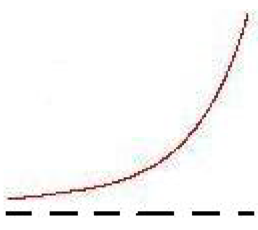
- The graph has a distinct concavity, which, depending on the value of ***b*** or a transformation, can be concave up or down.
- There is always a y-intercept.
- There is at most one x-intercept. This means that some graphs may have no x-intercept, while others may have one.

Problem 1:

Find the following for $f(x) = 2^x$.

a. Domain
b. Coordinates of the x-intercept
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:



- Domain:

Its domain consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Coordinates of the x-intercept:

$0 = 2^x$

$\log 0 = \log 2^x$

But any logarithm of ***0*** is undefined! Therefore, we can conclude that this function has **NO** x-intercept.

- Coordinates of the y-intercept:

$f(0) = 2^0 = 1$

The coordinates are ***(0, 1)***

NOTE: $a^0 = 1$, $a \neq 0$

- Equation of the Horizontal Asymptote:

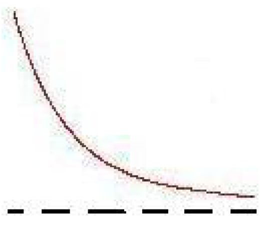
Since our equation is of the form $y = b^x$ with ***b = 2***, the x-axis is the horizontal asymptote whose equation is ***y = 0***.

Problem 2:

Find the following for $g(x) = (\frac{1}{2})^x$.

a. Domain
b. Coordinates of the x-intercept
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:



NOTE: This function can be changed to the form $g(x) = (\frac{1}{2})^x = (2^{-1})^x = 2^{-x}$. Now we can see that it is actually a reflection of the function $f(x) = 2^x$ about the y-axis.

- Domain:

Its domain consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Coordinates of the x-intercept:

$0 = (\frac{1}{2})^x$

$\log 0 = \log(\frac{1}{2})^x$

But any logarithm of 0 is undefined! Therefore, we can conclude that this function has **NO** x-intercept.

- Coordinates of the y-intercept:

$g(0) = (\frac{1}{2})^0 = 1$

The coordinates are ***(0, 1)***

- Equation of the Horizontal Asymptote:

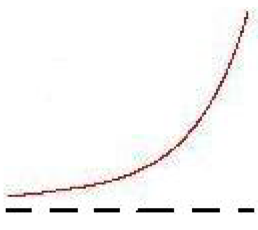
This function is of the form $y = b^x$. In this case $b = \frac{1}{2}$. Only vertical shifts of $y = b^x$ affect the location of the horizontal asymptote. Reflections **DO NOT** affect it. Therefore, the equation of the horizontal asymptote is still ***y = 0***.

Problem 3:

Find the following for $k(x) = 2^{x+1} - 5$.

a. Domain
b. Coordinates of the x-intercept. Round to 2 decimal places.
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:



- Domain:

Its domain consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Coordinates of x-intercept rounded to 2 decimal places

$$0 = 2^{x+1} - 5$$

Let's solve this exponential equation as usual.

$$5 = 2^{x+1}$$
$$\ln 5 = \ln 2^{x+1}$$
$$\ln 5 = (x+1)\ln 2$$
$$\frac{\ln 5}{\ln 2} = x + 1$$
$$x = \frac{\ln 5}{\ln 2} - 1$$

$$x \approx 1.32$$

The coordinates of the x-intercepts are approximately ***(1.32, 0)***.

- Coordinates of the y-intercept:

$$k(0) = 2^{0+1} - 5 = 2^1 - 5 = -3$$

The coordinates are ***(0, -3)***.

- Equation of the Horizontal Asymptote:

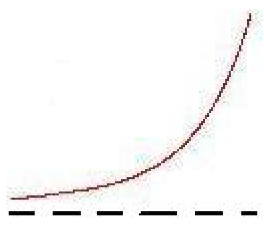
  This is a horizontal shift of $f(x) = 2^x$ by ***1*** unit to the left and a vertical shift of ***5*** units down. Since only vertical shifts affect the location of the horizontal asymptote, and we do have a vertical shift ***5*** units down, the equation of the horizontal asymptote becomes ***y = -5***.

Problem 4:

Find the following for $k(x) = e^x$.

a. Domain
b. Coordinates of the x-intercept
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:



- Domain:

  Its domain consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Coordinates of the x-intercept:

  $0 = e^x$

  $\ln 0 = \ln e^x$

  But any logarithm of ***0*** is undefined! Therefore, we can conclude that this function has **NO** x-intercept.

- Coordinates of the y-intercept:

  $k(0) = e^0 = 1$

  The coordinates are ***(0, 1)***

- Equation of the Horizontal Asymptote:

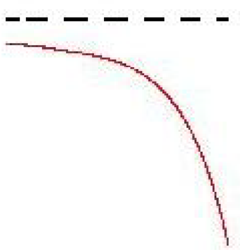
  Since our equation is of the form $y = b^x$ with ***b = e***, the x-axis is the horizontal asymptote whose equation is ***y = 0***.

Problem 5:

Find the following for $g(x) = -8e^{3x-4} + 16$.

a. Domain
b. Coordinates of the x-intercept. Round to 2 decimal places.
c. Coordinates of the y-intercept. Round to 2 decimal places.
d. Equation of the horizontal asymptote

The graph has the following shape:



- Domain:

Its domain consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Coordinates of the x-intercept:

$$0 = -8e^{3x-4} + 16$$
$$-16 = -8e^{3x-4}$$

Now we have to isolate the exponential expression by dividing both sides by ***-8***.

$$2 = e^{3x-4}$$

and finally, we can apply the natural logarithm to both sides as follows:

$$\ln 2 = \ln e^{3x-4}$$
$$\ln 2 = (3x-4)\ln e$$
$$\ln 2 = 3x - 4$$
$$\frac{\ln 2 + 4}{3} = x$$

$$x \approx 1.5644$$

The coordinates of the x-intercepts are ***(1.56, 0)*** rounded to 2 decimal places.

- Coordinates of the y-intercept:

$$g(0) = -8e^{3(0)-4} + 16$$
$$g(0) = -8e^{-4} + 16 \approx 15.8535$$

The coordinates of the y-intercepts are ***(0, 15.85)*** rounded to 2 decimal places.

- Equation of the Horizontal Asymptote:

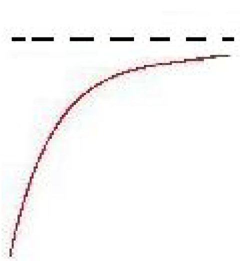
This is a complex transformation of $k(x) = e^{x}$. Since only vertical shifts affect the location of the horizontal asymptote, and we do have a vertical shift ***16*** units up, the equation of the horizontal asymptote becomes ***y = 16***.

Problem 6:

Find the following for $g(x) = -3e^{-6-2x} + 2$.

a. Domain
b. Coordinates of the x-intercept. Round to 2 decimal places.
c. Coordinates of the y-intercept. Round to 2 decimal places.
d. Equation of the horizontal asymptote

The graph has the following shape:



- Domain:

Its domain consists of ***All Real Numbers*** or $(-\infty, \infty)$ in *Interval Notation*.

- Coordinates of the x-intercept:

$$0 = -3e^{-6-2x} + 2$$
$$-2 = -3e^{-6-2x}$$

Now we have to isolate the exponential expression by dividing both sides by ***-3***.

$$\tfrac{2}{3} = e^{-6-2x}$$

and finally, we can apply the natural logarithm to both sides as follows:

$$\ln\tfrac{2}{3} = \ln e^{-6-2x}$$
$$\ln\tfrac{2}{3} = (-6-2x)\ln e$$
$$\ln\tfrac{2}{3} = -6-2x$$
$$\frac{\ln\tfrac{2}{3}+6}{-2} = x$$

$$x \approx -2.7973$$

The coordinates of the x-intercepts are ***(-2.80, 0)*** rounded to 2 decimal places.

- Coordinates of the y-intercept:

$$g(0) = -3e^{-6-2(0)} + 2$$
$$g(0) = -3e^{-6} + 2 \approx 1.9926$$

The coordinates of the y-intercepts are ***(0, 1.99)*** rounded to 2 decimal places.

- Equation of the Horizontal Asymptote:

This is a complex transformation of $k(x) = e^{x}$. Since only vertical shifts affect the location of the horizontal asymptotes, and we do have a vertical shift ***2*** units up, the equation of the horizontal asymptote becomes ***y = 2***.