DETAILED SOLUTIONS AND CONCEPTS - COMBINING FUNCTIONS

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PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definitions: Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions:

Sum Function: $f + g$

Difference Function: $f - g$

Product Function: $f \cdot g$

Quotient Function: $f \div g$, where $g \neq 0$

The **domain** of each of these functions is the intersection of the domains of f and g. Any x-value that makes the y-value undefined or imaginary must be excluded from the domain.

One other method developed to form new functions is called **composition**. Function compositions are often used to describe the physical world.

Definition of Function Composition

Given two functions, say f and g, the composition of f and g, denoted by $(f \circ g)(x)$, is defined by

$$(f \circ g)(x) = f[g(x)]$$

We read this as ***f composed of g*** equals ***f of g of x***.

The domain of $f \circ g$ can be determined by examining its final form. However, any numbers that are excluded from the domain of g must also be excluded from the domain of $f \circ g$.

Problem 1:

Given two functions $h(x) = x^2 + 3$ and $k(x) = 2x - 1$, find the following:

a. $(h + k)(x)$
b. $(h - k)(x)$
c. $(h \cdot k)(x)$
d. $(h \div k)(x)$
e. $(h \circ k)(x) = h[k(x)]$
f. $(k \circ h)(x) = k[h(x)]$

(a) $(h + k)(x) = x^2 + 3 + 2x - 1$

$= x^2 + 2x + 2$

The domain is the intersection of the domains of h and k. Since both functions have a domain of $(-\infty, \infty)$, the sum of these two functions also has a domain of $(-\infty, \infty)$.

(b) $(h - k)(x) = x^2 + 3 - (2x - 1)$

$= x^2 + 3 - 2x + 1$

$= x^2 - 2x + 4$

The domain is the intersection of the domains of h and k. Since both functions have a domain of $(-\infty, \infty)$, the difference of these two functions also has a domain of $(-\infty, \infty)$.

(c) $(h \cdot k)(x) = (x^2 + 3)(2x - 1)$

$= 2x^3 - x^2 + 6x - 3$

The domain is the intersection of the domains of h and k. Since both functions have a domain of $(-\infty, \infty)$, the product of these two functions also has a domain of $(-\infty, \infty)$.

(d) $\left(\frac{h}{k}\right)(x) = \frac{x^2+3}{2x-1}$

The domain is the intersection of the domains of h and k. Both functions have a domain consisting of all real numbers. However, for the quotient we must exclude ½ from the domain because it makes the denominator undefined. Therefore, the quotient of these two functions has a domain of $\{x \mid x \neq \frac{1}{2}\}$. That is, the domain includes all real numbers except ½

(e) $(h \circ k)(x) = h[k(x)]$

$$h(2x-1) = (2x-1)^2 + 3$$
$$= 4x^2 - 4x + 4$$

That is $(h \circ k)(x) = 4x^2 - 4x + 4$

The domain of the final function is $(-\infty, \infty)$. Since the domain of k is also $(-\infty, \infty)$, the final domain stays $(-\infty, \infty)$.

(f) $(k \circ h)(x) = k[h(x)]$

$$k(x^2+3) = 2(x^2+3) - 1$$
$$= 2x^2 + 5$$

That is, $(k \circ h)(x) = 2x^2 + 5$

The domain of the final function is $(-\infty, \infty)$. Since the domain of h is also $(-\infty, \infty)$, the final domain stays $(-\infty, \infty)$.

Problem 2:

Given two functions $f(x) = \sqrt{4-x}$ and $g(x) = x - 3$, find the following:

a. $(f \circ g)(x) = f[g(x)]$
b. $(g \circ f)(x) = g[f(x)]$

(a) $(f \circ g)(x) = f[g(x)]$

$$f(x-3) = \sqrt{4-(x-3)}$$
$$= \sqrt{7-x}$$

Thus, $(f \circ g)(x) = \sqrt{7-x}$

The domain of the final function is $(-\infty, 7]$. Since the domain of g is $(-\infty, \infty)$, the final domain stays $(-\infty, 7]$.

(b) $(g \circ f)(x) = g[f(x)]$

$g(\sqrt{4-x}) = \sqrt{4-x} - 3$

Thus, $(g \circ f)(x) = \sqrt{4-x} - 3$

The domain of the final function is $(-\infty, 4]$. Since the domain of f is also $(-\infty, 4]$, the final domain stays $(-\infty, 4]$.

Problem 3:

Given two functions $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$, find the following:

a. $(f \circ g)(x) = f[g(x)]$
b. $(g \circ f)(x) = g[f(x)]$
c. $(f \circ f)(x) = f[f(x)]$
d. $(g \circ g)(x) = g[g(x)]$

(a) $(f \circ g)(x) = f[g(x)]$

$$\begin{aligned} f(\tfrac{1}{3}x + \tfrac{2}{3}) &= 3(\tfrac{1}{3}x + \tfrac{2}{3}) - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

Hence $(f \circ g)(x) = x$

The domain of the final function is $(-\infty, \infty)$. Since the domain of g is also $(-\infty, \infty)$, the final domain stays $(-\infty, \infty)$.

(b) $(g \circ f)(x) = g[f(x)]$

$$\begin{aligned} g(3x - 2) &= \tfrac{1}{3}(3x - 2) + \tfrac{2}{3} \\ &= x - \tfrac{2}{3} + \tfrac{2}{3} \\ &= x \end{aligned}$$

Thus, $(g \circ f)(x) = x$

The domain of the final function is $(-\infty, \infty)$. Since the domain of f is also $(-\infty, \infty)$, the final domain stays $(-\infty, \infty)$.

(c) $(f \circ f)(x) = f[f(x)]$

$$f(3x-2) = 3(3x-2)-2$$
$$= 9x-6-2$$
$$= 9x-8$$

$$(f \circ f)(x) = 9x-8$$

The domain of the final function is $(-\infty, \infty)$. Since the domain of f is also $(-\infty, \infty)$, the final domain stays $(-\infty, \infty)$.

(d) $(g \circ g)(x) = g[g(x)]$

$$g(\tfrac{1}{3}x+\tfrac{2}{3}) = \tfrac{1}{3}(\tfrac{1}{3}x+\tfrac{2}{3})+\tfrac{2}{3}$$
$$= \tfrac{1}{9}x+\tfrac{2}{9}+\tfrac{2}{3}$$
$$= \tfrac{1}{9}x+\tfrac{8}{9}$$

$$(g \circ g)(x) = \tfrac{1}{9}x+\tfrac{8}{9}$$

The domain of the final function is $(-\infty, \infty)$. Since the domain of g is also $(-\infty, \infty)$, the final domain stays $(-\infty, \infty)$.

Problem 4:

The number n of cars produced by some factory in one day after t hours of operation is given by $n = 1000t - 10t^2$. If the cost C in dollars of producing n cars is $C(n) = 16000 + 400n$, find the cost C as a function of the time t of operating the factory.

$$C(n) = C(1000t - 10t^2) = 16000 + 400(1000t - 10t^2) = C(t)$$

and $C(t) = 16000 + 400000t - 4000t^2$ represents the cost as a function of time.

Problem 5:

The price ***p*** of some product and the quantity ***x*** sold obey the (demand) equation $p = -x + 30$ and the cost ***C*** of producing ***x*** units is $C = \frac{x^2 + 12000}{20}$. Find the cost ***C*** as a function of the price ***p***.

In this case, we must first solve the demand equation for ***x*** and then form the composite function ***C(x) = C(p)***.

$$p = -x + 30$$
$$x = 30 - p$$

Then $$C(x) = C(30-p) = \frac{(30-p)^2 + 12000}{20} = C(p)$$

and $C(p) = \frac{1}{20}p^2 - 3p + 645$ represents the cost as a function of the price.

Problem 6:

The surface area ***S*** of a spherical hot-air balloon is given by $S(r) = 4\pi r^2$, where ***r*** is the radius of the balloon. If the radius ***r*** increases with time ***t*** (in seconds) according to the formula $r = \frac{1}{2}t^3$, find the surface area ***S*** of the balloon as a function of the time ***t***.

$$S(r) = S(\tfrac{1}{2}t^3) = 4\pi(\tfrac{1}{2}t^3)^2 = S(t)$$

and $S(t) = \pi t^6$ represents the surface area of the balloon as a function of time.