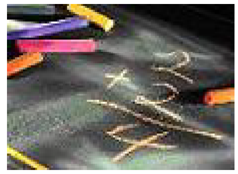

DETAILED SOLUTIONS AND CONCEPTS - INTRODUCTION TO ANGLES

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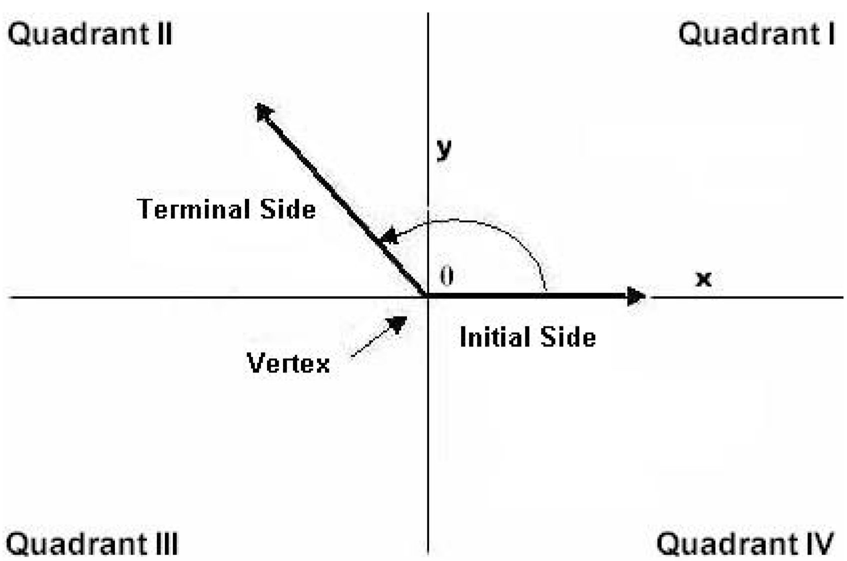
Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

PLEASE NOTE THAT YOU CANNOT ALWAYS USE A CALCULATOR ON THE ACCUPLACER - COLLEGE-LEVEL MATHEMATICS TEST! YOU MUST BE ABLE TO DO SOME PROBLEMS WITHOUT A CALCULATOR!

Definition of an Angle

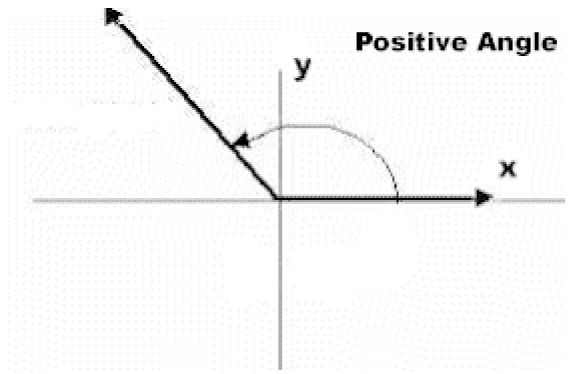
An angle is determined by rotating a ray about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side** of the angle. The point where the initial and the terminal side meet is called the **vertex** of the angle.

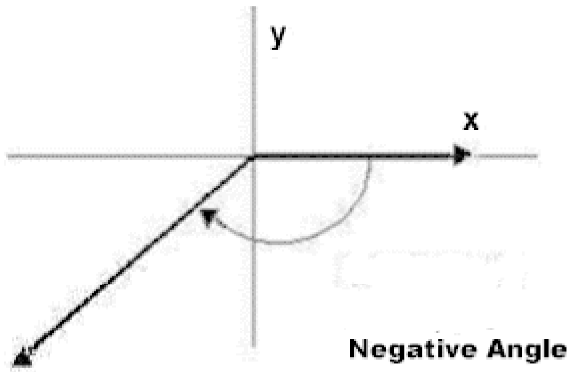
In trigonometry, we usually place an angle into a *Cartesian Coordinate System* with the initial side along the positive x-axis. We then say that the angle is in **standard position**.



- Angles with their terminal side in Quadrant I are called first-quadrant angles.
- Angles with their terminal side in Quadrant II are called second-quadrant angles.
- Angles with their terminal side in Quadrant III are called third-quadrant angles.
- Angles with their terminal side in Quadrant IV are called fourth-quadrant angles.

Angles can be of positive as well as negative magnitude. **Positive angles** are indicated by a counterclockwise rotation, and **negative angles** by a clockwise rotation.





Please note that as a purely numerical statement $-$ ***200º*** is less than $-$ ***100º***. However, since one often cares more about the magnitude of the angle rather than its orientation relative to a given line, some individuals in applied areas may state that $-$ ***200º*** is a larger angle measure than $-$ ***100º***.

However, this is a slightly "sloppy" use of the terminology (but probably the generally accepted "standard abuse" of it) because one should really say that $-$ ***200º*** is an angle of larger magnitude than $-$ ***100º***.

Angle Measurement

Degrees, Minutes, and Seconds

Most commonly, angles are measured in degrees. This is indicted by the degree sign **º** next to a number. For example, ***45º***.

Degrees can further be divided into minutes (**'**) and seconds (**"**). That is,

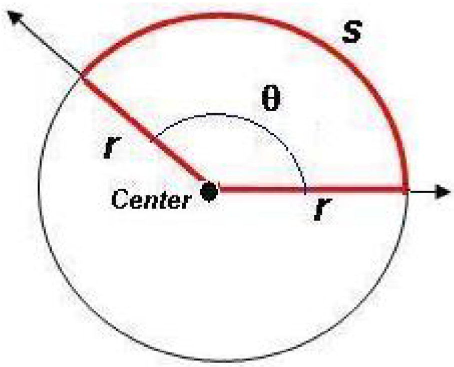
1º = 60' (minutes) **using the apostrophe on the computer keyboard**

1' = 60" (seconds) **using the quotation mark on the keyboard**

Most calculators convert from decimal degrees (e.g., ***18.5º***) to degrees, minutes, and seconds (e.g., ***18º 30'***) and vice versa. Consult the User's Guide of your calculator!

Radians

Place the vertex of an angle θ at the center of a circle with radius ***r***. Let ***s*** be the length of the arc opposite θ on the circumference of the circle. See picture below.

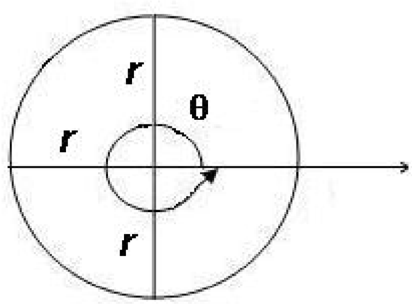


Then the **radian measure of** θ is defined to be $\boldsymbol{s \div r}$.

Unlike degrees, which use the symbol º, radians do NOT use a symbol nor will the measure be followed by the word "radians". Later, in context, you will know when you are working with radian measure.

Relationship between Degree and Radian Measure

We will now compare degree measure and radian measure given one complete rotation of the central angle θ.



The magnitude of angle θ in degrees is ***360º***. However, the magnitude of angle θ in radians is

$$\frac{s}{r} = \frac{2\pi r}{r} = 2\pi$$

Subsequently, we will say that ***360º*** is equivalent to 2π.

- Unlike degrees, which uses the symbol º, radians do NOT use a symbol nor will they be followed by the word "radians". Later, in context, you will know when you are working with radians!
- Whenever possible, express radians in terms of π. However, do NOT assume that every radian measure must contain the symbol π.
- Angles expressed in radians are usually given the same names as angles expressed in degrees.
- The definitions for *Right, Straight, Quadrantal, Coterminal*, and *Reference Angles* still apply, except that the angles are now expressed in radians.
- Just like degrees, radians can be positive and negative.

Please note:

If $360° \equiv 2\pi$ (radians)

then

$$1° \equiv \frac{\pi}{180}\text{(radians)}$$

and using basic algebra, we find that

$$1\text{(radian)} \equiv \left(\frac{180}{\pi}\right)^{\circ}$$

You must memorize these conversions for this course!

NOTE: The symbol $\equiv$ means "is equivalent to", which is a better term than "equal" in this case.

Common Names of Angles

Theta: θ Alpha: α Beta: β Gamma: γ

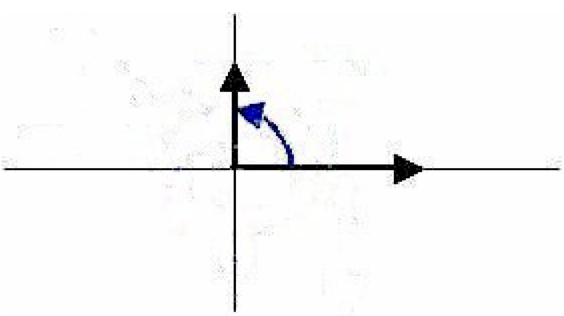
Sometimes, angles are simply referred to as angle x or angle y.

Acute Angles

Angles whose magnitude is greater than 0^o but less than 90^o.

Right Angles

Angles whose magnitude is exactly 90^o.

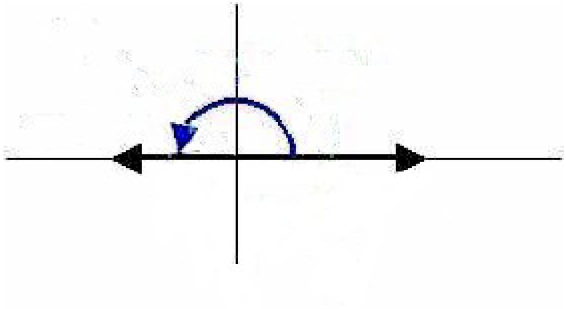


Obtuse Angles

Angles whose magnitude is greater than 90^o but less than 180^o.

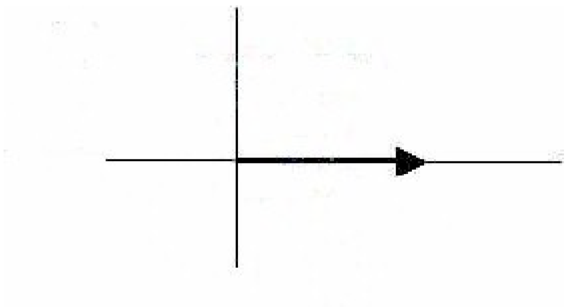
Straight Angles

Angles whose magnitude is exactly ***180º***.



Zero-Degree Angle

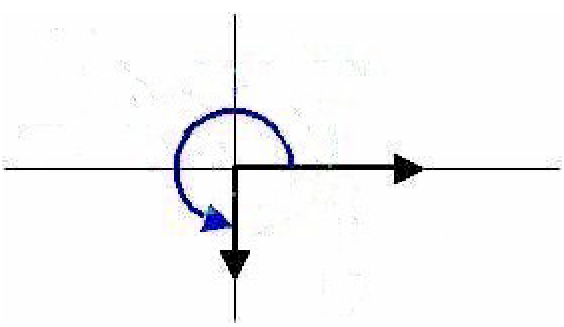
Angles whose magnitude is ***0º***.



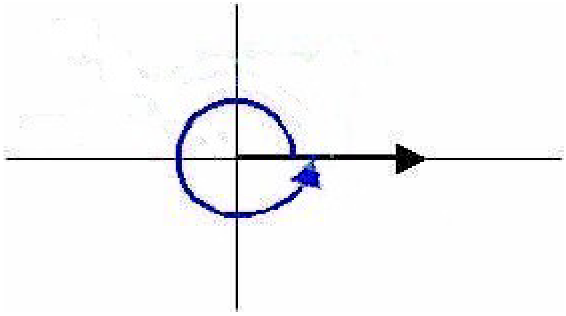
NOTE: The terminal side and the initial side are the same. There is no arrow indicating direction! Compare to an angle of measure ***360º*** !

Angles with Magnitude larger than *180º* - Some Examples!

270º

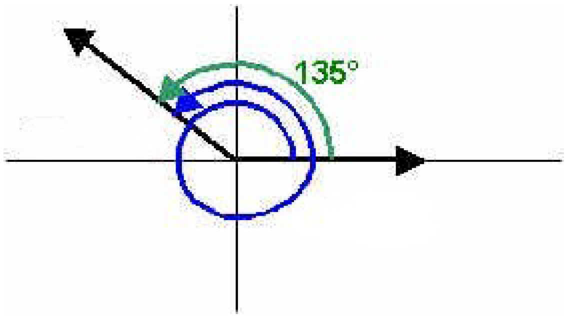


360^o



NOTE: Unlike with the ***0^o*** angle, there is an arrow indicating direction!

$495^o = 360^o + 135^o$



Quadrantal Angles

When the terminal side of an angle lies along a coordinate axis, the angle is called a *Quadrantal Angle*. For example, $\pm 0^o$, $\pm 90^o$, $\pm 180^o$, $\pm 270^o$, 360^o, $\pm 450^o$, etc. are magnitudes of *Quadrantal Angles*.

Any integer multiple of ***90^o*** is considered a *Quadrantal Angle*. Remember that integers are positive and negative whole numbers!!!

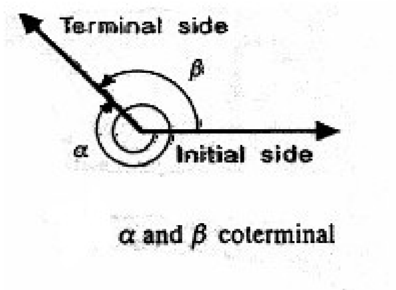
Coterminal Angles

Two angles are coterminal if they have the same initial and terminal sides. Any angle has infinitely many coterminal angles.

You can find an angle that is coterminal to a given angle by adding integer multiples of $360^\circ \equiv 2\pi$ to the given angle. Remember that integers include ***0*** and positive and negative whole numbers!

For instance, the angles of magnitude ***135^o*** and ***-585^o*** are coterminal.

$135^o + 360^o(-2) = -585^o$. See picture below.



Reference Angles

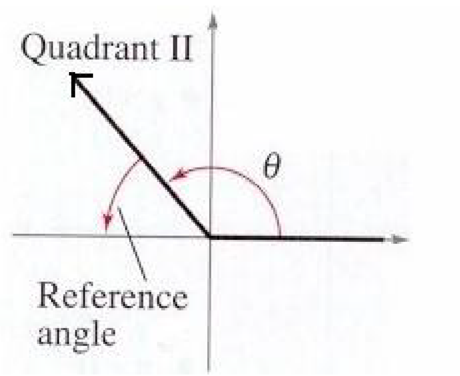
The *Reference Angle*, let's call it α, is the acute positive angle between the terminal side of an angle θ and the horizontal axis in a coordinate system.

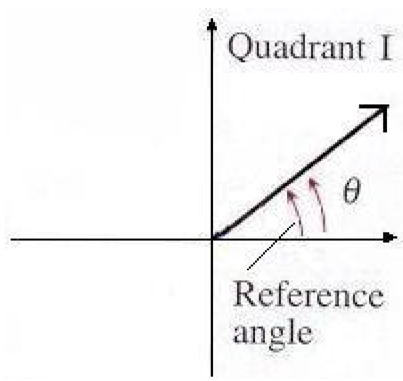
NOTES:

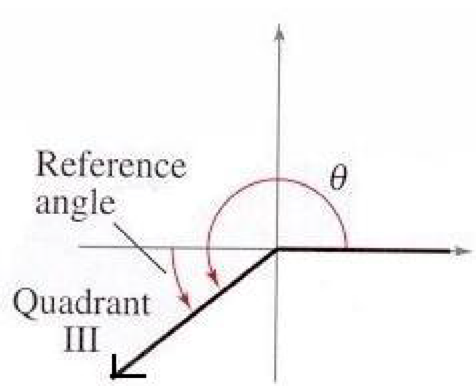
(1) *Quadrantal Angles* do NOT have *Reference Angles*.

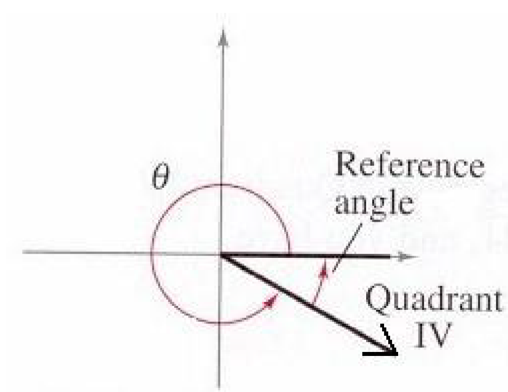
(2) Negative angles have the same *Reference Angles* as their positive counterparts.

(3) *Coterminal Angles* have the same *Reference Angle*.









If an angle θ is **positive and between *0º* and *360º***, you can calculate the magnitude of its reference angle α as follows:

θ is a first-quadrant angle: $\alpha = \theta$

θ is a second-quadrant angle: $\alpha = 180° - \theta$

θ is a third-quadrant angle: $\alpha = \theta - 180°$

θ is a fourth-quadrant angle: $\alpha = 360° - \theta$

Special Angles Occurring Frequently in Mathematics and the Sciences

0º, 30º, 45º, 60º, 90º, 180º, 270º, and ***360º***

Let's find their **EXACT** radian equivalents. **You must memorize these conversions for this course!**

Using the fact that $1° \equiv \frac{\pi}{180}$ (radians), we find

$0° \equiv 0$ (radians) $\quad 30° \equiv \frac{\pi}{6} \quad 45° \equiv \frac{\pi}{4} \quad 60° \equiv \frac{\pi}{3}$

$90° \equiv \frac{\pi}{2} \quad 180° \equiv \pi \quad 270° \equiv \frac{3\pi}{2} \quad 360° \equiv 2\pi$

NOTE: If possible, always try to use EXACT radians. For example, $\pi/6$ is an EXACT value. Do not use its approximations 0.52 or 0.17π.

Integer Multiples of Special Angles

Integer multiples of the special angles also occur frequently in Mathematics and the Sciences. We will use them often!

To change them to radians, you can multiply each angle by $\pi/180$ and then reduce to lowest terms!

Let's use the angle $210° = 7 \cdot 30°$ for example. We could change it to radians as follows.

$$210° \equiv 210 \cdot \frac{\pi}{180} = \frac{7\pi}{6} \text{ radians}$$

, etc.

However, integer multiples of special angles occur so frequently, that you must often convert then quickly to radians. The following method might be a little speedier.

For example, since we know that $30^o \equiv \pi/6$ and since $210^\circ = 7 \cdot 30^\circ$, we can say that

$$210^\circ \equiv 7 \cdot \frac{\pi}{6} = \frac{7\pi}{6}.$$

Radian Equivalents of Negative Angles

Just place a negative sign in front of the radian equivalent. For example,

$-60^\circ \equiv -\frac{\pi}{3}$ or $-45^\circ \equiv -\frac{\pi}{4}$

Problem 1:

Change $45^o\ 14'\ 39''$ (45 degrees and 14 minutes and 39 seconds) to decimal degree form. Round to two decimal places.

(1) Calculate $\left(45 + \frac{14}{60} + \frac{39}{3600}\right)$. Type the entire calculation into your calculator. Do not round until you have the final answer: $45^o\ 14'\ 39'' \approx 45.24^o$

(2) Use the angle-conversion feature of your calculator (see User's Guide).

Problem 2:

Change $-45^o\ 14'\ 39''$ to decimal degree form. Round to two decimal places.

(1) Calculate $-\left(45 + \frac{14}{60} + \frac{39}{3600}\right)$. Type the entire calculation into your calculator. Do not round until you have the final answer: $-45^o\ 14'\ 39'' \approx -45.24^o$

(2) Use the angle-conversion feature of your calculator (see User's Guide).

Problem 3:

Change 84.78^o to degrees, minutes, and seconds rounded to whole numbers.

(1) Take $.78$ away from 84.78 and calculate the minutes as follows: $.78(60)' = 46.8'$

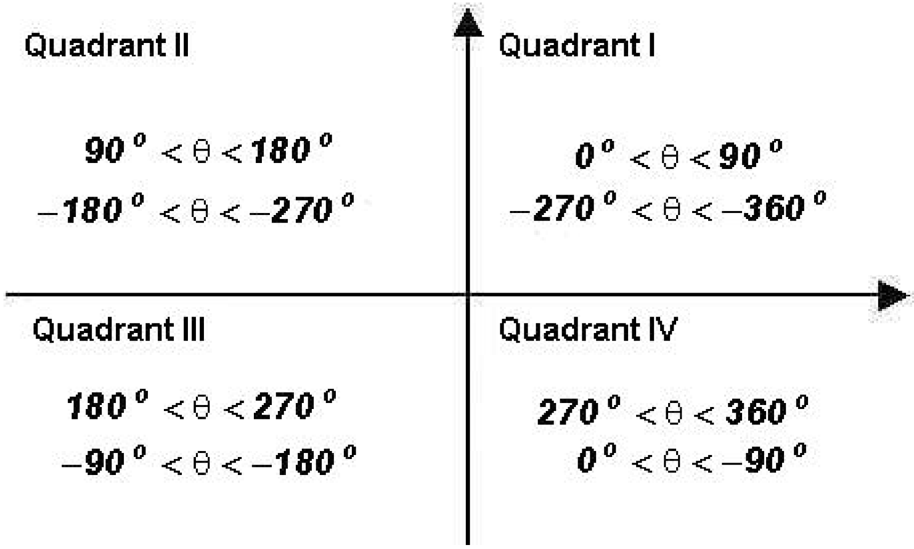
Take $.8$ away from 46.8 and calculate the seconds as follows: $.8(60)''$

Thus, $84.78^o = 84^o\ 46'\ 48''$

(2) Use the angle-conversion feature of your calculator (see User's Guide).

Problem 4:

Find the location of the terminal side of the following angles in degrees utilizing the chart below.



NOTE: To determine the location of the terminal side of an angle that is larger than *360º*, find the largest multiple of positive or negative *360º* contained in the angle measurement. Use the difference between it and the angle measurement to determine the location of the terminal side of the angle.

a. 57^o
Quadrant I angle

b. -57^o
Quadrant IV angle

c. 145^o
Quadrant II angle

d. -145^o
Quadrant III angle

e. 236^o
Quadrant III angle

f. 315^o
Quadrant IV angle

g. 495^o
$495^o = 360^o + 135^o$ - Quadrant II angle using the angle 135^o, which is coterminal to the angle 495^o.

h. -396^o
$-396^o = -360^o + (-36^o)$ - Quadrant IV angle using the angle -36^o, which is coterminal to the angle -396^o.

Problem 5:

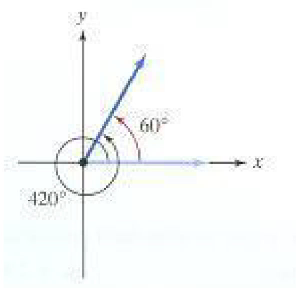
Find a positive angle that is smaller than 360^o and is coterminal with angle 420^o.

Remember that you can find an angle that is coterminal to a given angle by adding integer multiples of $360^\circ \equiv 2\pi$ to the given angle.

This means that we must add enough integer multiples of $360^\circ \equiv 2\pi$ to the given angle to end up with a positive angle that is smaller than 360^o.

That is, $420^o + 360^o(-1) = 60^o$ Note that we used a negative integer!

We find that a 60^o angle is coterminal with a 420^o angle.



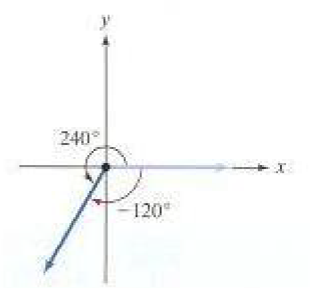
Problem 6:

Find a positive angle that is smaller than 360^o and is coterminal with a -120^o angle.

This means that we must add enough integer multiples of $360^{\circ} \equiv 2\pi$ to the given angle to end up with a positive angle that is smaller than 360^o.

That is, $-120^o + 360^o(1) = 240^o$

We find that a 240^o angle is coterminal with a -120^o angle.



Problem 7:

Find all angles that are coterminal with a 315^o angle.

$315^o + 360^o k$, where k is any integer

For example,

when $k = 2$, then $315^o + 360^o(2) = 1035^o$

when $k = 1$, then $315^o + 360^o(1) = 675^o$

when $k = 0$, then $315^o + 360^o(0) = 315^o$

when $k = -1$, then $315^o + 360^o(-1) = -45^o$

when $k = -2$, then $315^o + 360^o(-2) = -405^o$

Problem 8:

Find the reference angle for a ***57°*** angle.

Since we are dealing with a first-quadrant angle, the reference angle is ***57°***.

Problem 9:

Find the reference angle for a ***-57°*** angle.

Remember that negative angles have the same *Reference Angles* as their positive counterparts!

The reference angle for a ***-57°*** angle is the same as that for a ***57°*** angle. Therefore, the reference angle is ***57°***.

Problem 10:

Find the reference angle for a ***145°*** angle.

Since we are dealing with a second-quadrant angle, the reference angle is ***180° - 145° = 35°***.

Problem 11:

Find the reference angle for a ***-145°*** angle.

Remember that negative angles have the same *Reference Angles* as their positive counterparts!

The reference angle for a ***-145°*** angle is the same as that for a ***145°*** angle. Therefore, the reference angle is ***35°***.

Problem 12:

Find the reference angle for a ***236°*** angle.

Since we are dealing with a third-quadrant angle, the reference angle is ***236° - 180° = 56°***.

Problem 13:

Find the reference angle for a ***315°*** angle.

Since we are dealing with a fourth-quadrant angle, the reference angle is ***360° - 315° = 45°***.

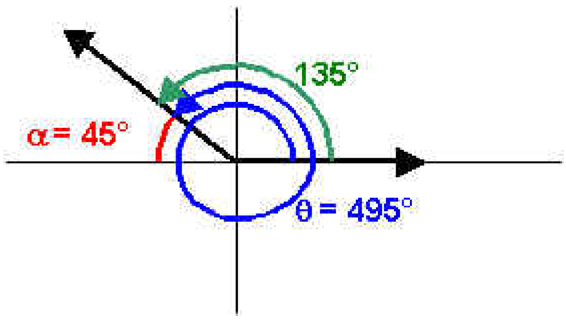
Problem 14:

Find the reference angle for a 495^o angle.

Remember that coterminal angles have the same *Reference Angle* !

Since $495^o = 360^o + 135^o$, we use the 135^o angle (which is coterminal with the 495^o angle) to find the reference angle.

The 135^o angle is a second-quadrant angle with a reference angle of $180^o - 135^o = 45^o$. This is also the reference angle of the 495^o angle.



Problem 15:

Find the reference angle for a -396^o angle.

Remember that negative angles have the same *Reference Angles* as their positive counterparts!

The reference angle for the -396^o angle is the same as the reference angle for the 396^o angle.

Remember that coterminal angles have the same *Reference Angle* !

Since $396^o = 360^o + 36^o$, we use the 36^o angle (which is coterminal with the 396^o angle) to find the measure of the reference angle.

The 36^o angle is a first-quadrant angle with a reference angle of 36^o. This is also the reference angle the for angles -396^o and 396^o.

Problem 16:

Express 315^o in EXACT radians reduced to lowest terms.

This is a multiple of a special angle, therefore, we can use the following shortcut:

$$315^\circ = 7 \cdot 45^\circ \equiv 7 \cdot \frac{\pi}{4} = \frac{7\pi}{4}$$

OR given $1^\circ \equiv \frac{\pi}{180}$ (radians)

$$315^\circ \equiv 315 \cdot \frac{\pi}{180} = \frac{7\pi}{4}$$

Please note that $7\pi/4 \approx 5.498$. The measure $7\pi/4$ is considered **EXACT** and 5.498 is its decimal approximation. Also, do NOT write 1.75π when asked for an **EXACT** measure!

Problem 17:

Express 330^o in EXACT radians reduced to lowest terms.

This is a multiple of a special angle, therefore, $330^\circ = 11 \cdot 30^\circ \equiv 11 \cdot \frac{\pi}{6} = \frac{11\pi}{6}$

Problem 18:

Express 120^o in EXACT radians reduced to lowest terms.

This is a multiple of a special angle, therefore, $120^\circ = 2 \cdot 60^\circ \equiv 2 \cdot \frac{\pi}{3} = \frac{2\pi}{3}$

Problem 19:

Express 164^o in EXACT radians reduced to lowest terms.

This is NOT a multiple of a special angle, therefore, $164^\circ \equiv 164\left(\frac{\pi}{180}\right) = \frac{41\pi}{45}$

Problem 20:

Express -46.52^o in radians rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

NOTE: Whenever possible and practical, use the π key on your calculator instead of 3.14 because this will result in more exact calculations.

$$-46.52^\circ \equiv -46.52 \cdot \frac{\pi}{180} \approx -0.81$$

Problem 21:

Express the radian measure $-\dfrac{4\pi}{3}$ in EXACT degree measure reduced to lowest terms.

This is a multiple of a special angle, therefore, we can use the following short-cut:

$$-\frac{4\pi}{3} = -4\cdot\frac{\pi}{3} \equiv -4\cdot 60^\circ = -240^\circ$$

OR given $1\text{ (radian)} \equiv \left(\frac{180}{\pi}\right)^\circ$

$$-\frac{4\pi}{3} \equiv -\frac{4\pi}{3}\cdot\frac{180^\circ}{\pi} = -240^\circ$$

Problem 22:

Express the radian measure $\dfrac{11\pi}{36}$ in EXACT degree measure reduced to lowest terms.

Since this is NOT a multiple of a special angle, you can only find its degree equivalent as follows:

$$\frac{11\pi}{36} \equiv \frac{11\pi}{36}\left(\frac{180}{\pi}\right)^\circ = 55^\circ$$

Problem 23:

Express the radian measure $\dfrac{6\pi}{7}$ in degree measure rounded to two decimal places. **Use the π key on your calculator instead of *3.14*.**

$$\frac{6\pi}{7} \equiv \frac{6\pi}{7}\left(\frac{180}{\pi}\right)^\circ \approx 154.29^\circ$$

Problem 24:

Express the radian measure 4.8 in degree measure rounded to two decimal places. **Use the π key on your calculator instead of *3.14*.**

$$4.8 \equiv 4.8\left(\frac{180}{\pi}\right)^\circ \approx 275.02^\circ$$

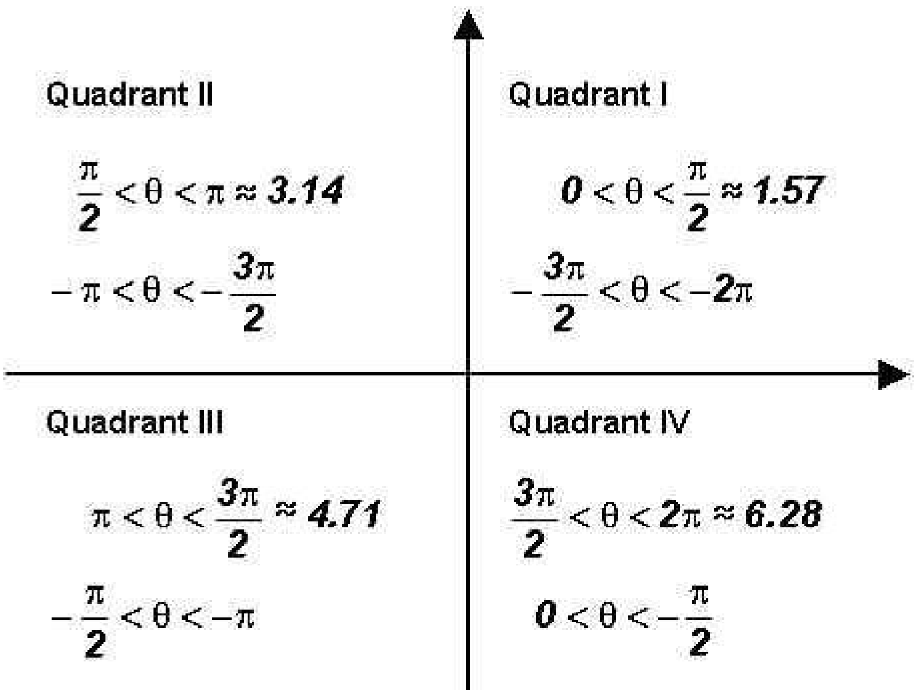
Problem 25:

Express the radian measure $\boldsymbol{5}$ in degree measure rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

$$5 \equiv 5\left(\frac{180}{\pi}\right)^{\circ} \approx 286.48^{\circ}$$

Problem 26:

Find the location of the terminal side of the following angles in radians utilizing the chart below.



NOTE: To determine the location of the terminal side of an angle that is larger than 2π, find the largest multiple of positive or negative 2π contained in the angle measure. Use the difference between it and the angle measure to determine the location of the terminal side of the angle.

a. 1.3
Quadrant I angle

b. $\pi/3$
Quadrant I angle

c. 2.7
Quadrant II angle

d. $3\pi/4$
Quadrant II angle

e. 4.2
Quadrant III angle

f. $7\pi/6$
Quadrant III angle

g. 5.3
Quadrant IV angle

h. $5\pi/3$
Quadrant IV angle

i. -1.3
Quadrant IV angle

j. $-\pi/3$
Quadrant IV angle

k. $-7\pi/3$
$-7\pi/3 = -2\pi + (-\pi/3)$ **-** Quadrant IV angle using the angle $-\pi/3$, which is coterminal to the angle $-7\pi/3$.

l. 8.98
$8.98 \approx 2\pi + 2.7$ - Quadrant II angle using the angle 2.7, which is coterminal to the angle 8.98.

Problem 27:

Find a positive angle in radians that is smaller than 2π and is coterminal with an angle of radian measure $7\pi/3$.

Remember that you can find an angle that is coterminal to a given angle by adding integer multiples of $360^o \equiv 2\pi$ (radians) to the given angle.

This means that we must add enough integer multiples of $360° \equiv 2\pi$ to the given angle to end up with a positive angle that is smaller than 2π.

That is, $7\pi/3 + 2\pi(-1) = \pi/3$ Note that we used a negative integer!

We find that the angle $\pi/3$ is coterminal with the angle $7\pi/3$.

Problem 28:

Find a positive angle in radians that is smaller than 2π and is coterminal with an angle of radian measure $-2\pi/3$.

Again, this means that we must add enough integer multiples of $360° \equiv 2\pi$ to the given angle to end up with a positive angle that is smaller than 2π.

That is, $-2\pi/3 + 2\pi(1) = 4\pi/3$

We find that the angle $4\pi/3$ is coterminal with the angle $-2\pi/3$.

Problem 29:

Find all angles that are coterminal with the angle $7\pi/4$.

$7\pi/4 + 2\pi k$ **,** where k is any integer

For example,

when $k = 2$, then $7\pi/4 + 2\pi(2) = 23\pi/4$

when $k = 1$, then $7\pi/4 + 2\pi(1) = 15\pi/4$

when $k = 0$, then $7\pi/4 + 2\pi(0) = 7\pi/4$

when $k = -1$, then $7\pi/4 + 2\pi(-1) = -\pi/4$

when $k = -2$, then $7\pi/4 + 2\pi(-2) = -9\pi/4$

Problem 30:

Find the reference angle in radians for an angle with radian measure 1.3.

Since we are dealing with a first-quadrant angle, the reference angle is 1.3.

Problem 31:

Find the reference angle in radians for an angle with radian measure -1.3.

Remember that negative angles have the same *Reference Angles* as their positive counterparts!

The reference angle for an angle of radian measure -1.3 is the same as the reference angle for an angle of radian measure 1.3. That is, the reference angle for -1.3 is 1.3.

Problem 32:

Find the EXACT reference angle in radians for an angle with radian measure $\pi/3$.

Since we are dealing with a first-quadrant angle, the reference angle is $\pi/3$.

Please note that $\pi/3 \approx 1.047$. The measure $\pi/3$ is considered **EXACT** and 1.047 is its decimal approximation. Also, do NOT write 0.33π.

Problem 33:

Find the reference angle for an angle with radian measure 2.7 rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

Since we are dealing with a second-quadrant angle, the reference angle is $\pi - 2.7 \approx 0.44$.

Problem 34:

Find the reference angle for an angle with radian measure -2.7 rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

The reference angle for an angle of radian measure -2.7 is the same as the reference angle for an angle of radian measure 2.7. That is, the reference angle for -2.7 is $\pi - 2.7 \approx 0.44$.

Remember that negative angles have the same *Reference Angles* as their positive counterparts!

Problem 35:

Find the EXACT reference angle in radians for an angle with radian measure $3\pi/4$.

Since we are dealing with a second-quadrant angle, the reference angle is $\pi - 3\pi/4 = \pi/4$.

Problem 36:

Find the reference angle in radians for an angle with radian measure 4.2 rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

Since we are dealing with a third-quadrant angle, the reference angle is $4.2 - \pi \approx 1.06$.

Problem 37:

Find the reference angle for an angle with radian measure 5.3 rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

Since we are dealing with a fourth-quadrant angle, the reference angle is $2\pi - 5.3 \approx 0.98$.

Problem 38:

Find the reference angle for an angle with radian measure 11.08 rounded to two decimal places. **Use the π key on your calculator instead of 3.14.**

Since $11.08 \approx 2\pi + 4.8$, we use 4.8 (which is coterminal with 11.08) to find the reference angle.

Remember that coterminal angles have the same *Reference Angle*!

4.8 is a fourth-quadrant angle with a reference angle of $2\pi - 4.8 \approx 1.48$. This is also the reference angle for the angle with radian measure 11.08.

Problem 39:

Find the EXACT reference angle in radians for an angle with radian measure $-9\pi/4$.

Remember that negative angles have the same *Reference Angles* as their positive counterparts!

Therefore, the reference angle for an angle with radian measure $-9\pi/4$ is the same as the reference angle for an angle with radian measure $9\pi/4$.

Remember that coterminal angles have the same *Reference Angle*!

Since $\boldsymbol{9\pi/4 = 2\pi + \pi/4}$, we use $\boldsymbol{\pi/4}$ (which is coterminal with $\boldsymbol{9\pi/4}$) to find the reference angle.

$\boldsymbol{\pi/4}$ is a first-quadrant angle with a reference angle of $\boldsymbol{\pi/4}$. This is also the reference angle for the angle with radian measure $\boldsymbol{-9\pi/4}$ and $\boldsymbol{9\pi/4}$.