$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$

DIFFERENTIATION RULES FOR SOME TRANSCENDENTAL FUNCTIONS

The exponential, logarithmic, trigonometric, and inverse trigonometric functions are called *transcendental functions.*

The Derivative of the Exponential Function:

You can find the proof of this rule in the online textbook as a separate document.

Let the base b be any real positive number, except for ***1***.

If $f(x) = b^x$, then $f'(x) = b^x \ln b$

Special Case (most frequently used) - The *Natural Exponential Function* with base *e*

If $f(x) = e^x$, then $f'(x) = e^x$ since $\ln e = 1$!

All calculators have an ***e-key***. Most likely it is labeled e^x. You MUST use the calculator key, and NOT ***2.718*** !!!

The Derivative of the Logarithmic Function:

The logarithm bases that occur most frequently in application problems are ***e*** and ***10***, where *e* is the famous irrational number *2.718281828459045...* , often rounded to ***2.72***.

$\log_{10} x$ is usually abbreviated as $\log x$. It is called the common logarithm!

$\log_e x$ is usually abbreviated as $\ln x$, pronounced "el en of x". It is called the natural logarithm!

Point of Interest: The number *e* was found while investigating what happens to the value of $(1 + \frac{1}{m})^m$ as *m* gets infinitely large. See *Proof of the Definition of the Number e* in the online textbook.

All calculators have a ***LOG-key*** (base 10) and an ***LN-key*** (base *e*). To evaluate other bases you must use the **Change of Base Property** as follows:

Given two logarithm bases, say *a* and *b*, we can state the following:

$$\log_b M = \frac{\log_a M}{\log_a b}$$

You can find the proof of the logarithm rule in the online textbook as a separate document.

Let the base ***b*** be any real positive number, except for ***1***.

If $f(x) = \log_b x$, then $f'(x) = \dfrac{1}{x \ln b}$

Special Case (most frequently used) - The *Natural Logarithm* with base *e*

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ since $\ln e = 1$!

Derivatives of the Trigonometric Functions:

You can find the proofs of these rules in the online textbook as a separate document.

If $f(x) = \sin x$, then $f'(x) = \cos x$

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$

If $f(x) = \sec x$, then $f'(x) = \sec x \tan x$

If $f(x) = \cos x$, then $f'(x) = -\sin x$

If $f(x) = \cot x$, then $f'(x) = -\csc^2 x$

If $f(x) = \csc x$, then $f'(x) = -\csc x \cot x$

Please note that all trigonometric functions starting with "co" have negative derivatives!

Problem 1:

For $f(x) = 7x \tan x$, find $f'(x)$.

Here we have a product with 3 factors. Since two of them contain variables, we'll assign the constant factor to the next variable factor and use the *Product Rule*.

That is, $u = 7x$, and $v = \tan x$,

then $u' = 7$ and $v' = \sec^2 x$

and $f'(x) = 7 \tan x + 7x \sec^2 x$

Problem 2:

Find the derivative of $g(t) = t^2 - t\sin t$.

Here we must use the *Sum/Difference Rule*, but a *Product Rule* is necessary for the second term.

That is, $u = t$, and $v = \sin t$,

then $u' = 1$ and $v' = \cos t$

and $g'(t) = 2t - (\sin t + t\cos t) = 2t - \sin t - t\cos t$

Problem 3:

Use the *Quotient Rule* to differentiate $y = \dfrac{1 - \csc x}{2x - 1}$.

That is, $u = 1 - \csc x$, and $v = 2x - 1$,

then $u' = -(-\csc x \cot x) = \csc x \cot x$ and $v' = 2$

$$\frac{dy}{dx} = \frac{\csc x \cot x(2x-1) - (1-\csc x)(2)}{(2x-1)^2}$$

and $\dfrac{dy}{dx} = \dfrac{2x\csc x \cot x - \csc x \cot x - 2 + 2\csc x}{(2x-1)^2}$

or, if you want , $\dfrac{dy}{dx} = \dfrac{\csc x(2x\cot x - \cot x + 2) - 2}{(2x-1)^2}$

It is NOT necessary to simplify this answer by attempting to use trigonometric identities! It is also standard procedure to leave the denominator in factored form.

Problem 4:

Differentiate $y = \dfrac{1}{\cos x \cot x}$ using the *Quotient Rule* and the *Product Rule*.

Quotient Rule:

$u = 1$ $\qquad u' = 0$ $\qquad v = \cos x \cot x$

Please note that we must use the *Product Rule* to find v'!

$$v' = -\sin x \cot x + \cos x(-\csc^2 x)$$
$$= -\sin x \cot x - \cos x \csc^2 x$$

$$\frac{dy}{dx} = \frac{0(\cos x \cot x) - (1)(-\sin x \cot x - \cos x \csc^2 x)}{(\cos x \cot x)^2}$$

and $\dfrac{dy}{dx} = \dfrac{\sin x \cot x + \cos x \csc^2 x}{\cos^2 x \cot^2 x}$

It is NOT necessary to simplify this answer by attempting to use trigonometric identities!

Product Rule:

In order to use the *Product Rule*, we must change

$y = \dfrac{1}{\cos x \cot x} = \dfrac{1}{\cos x} \cdot \dfrac{1}{\cot x}$ using *Reciprocal Identities* as follows:

Since $\dfrac{1}{\cos x} = \sec x$ and $\dfrac{1}{\cot x} = \tan x$,

then $y = \sec x \tan x$ which is a product.

To use the *Product Rule* we let $u = \sec x$, and $v = \tan x$. Then $u' = \sec x \tan x$ and $v' = \sec^2 x$

$$\frac{dy}{dx} = \sec x \tan x \tan x + \sec x \sec^2 x$$

and $\dfrac{dy}{dx} = \sec x \tan^2 x + \sec^3 x$

Please note that the answers derived from the *Quotient* and *Product Rule* seem to be totally different. However, they should not be. Let's use the identity verification process to prove that they are actually identical.

$$\frac{\sin x \cot x + \cos x \csc^2 x}{\cos^2 x \cot^2 x} \stackrel{?}{=} \sec x \tan^2 x + \sec^2 x$$

Let's try to change the left side! Remember that identity verification requires you to work on one side only!

$$\frac{\sin x \cot x + \cos x \csc^2 x}{\cos^2 x \cot^2 x} \stackrel{?}{=} \sec x \tan^2 x + \sec^2 x$$

Let's separate the fraction on the left as follows:

$$\frac{\sin x \cot x}{\cos^2 x \cot^2 x} + \frac{\cos x \csc^2 x}{\cos^2 x \cot^2 x} =$$

Since $\frac{1}{\cot^2 x} = \tan^2 x$ and $\frac{1}{\cos^2 x} = \sec^2 x$ which are needed on the right side, we'll preserve those immediately and then we'll try to change the remaining terms to $\sec x$.

$$\frac{\sin x \cot x}{\cos^2 x} \cdot \frac{1}{\cot^2 x} + \frac{1}{\cos^2 x} \cdot \frac{\cos x \csc^2 x}{\cot^2 x} =$$

$$\frac{\sin x \cot x}{\cos^2 x} \cdot \tan^2 x + \sec^2 x \cdot \frac{\cos x \csc^2 x}{\cot^2 x} =$$

Now we'll use *Reciprocal* and *Quotient Identities* as follows:

$$\frac{\sin x \frac{\cos x}{\sin x}}{\cos^2 x} \cdot \tan^2 x + \sec^2 x \cdot \frac{\cos x \frac{1}{\sin^2 x}}{\frac{\cos^2 x}{\sin^2 x}} =$$

Next we'll change the complex fraction in the second term into a single fraction. Remember that dividing by a fraction is the same as multiplying by the reciprocal of the fraction!

$$\frac{1}{\cos x} \cdot \tan^2 x + \sec^2 x \cdot \frac{\cos x}{\sin^2 x} \cdot \frac{\sin^2 x}{\cos^2 x} =$$

$$\frac{1}{\cos x} \cdot \tan^2 x + \sec^2 x \cdot \frac{1}{\cos x} =$$

$$\sec x \tan^2 x + \sec^2 \sec x =$$

$$\sec x \tan^2 x = \sec x \tan^2 x + \sec^2 x$$

Therefore, we have shown that the derivative found using the *Quotient Rule* equals the one found when using the *Product Rule*.

Problem 5:

Use the *Quotient Rule* to differentiate $f(x) = \frac{1 + \sec x}{1 - \sec x}$.

That is, $u = 1 + \sec x$, and $v = 1 - \sec x$.

Then $u' = \sec x \tan x$ and $v' = -\sec x \tan x$

$$\frac{dy}{dx} = \frac{\sec x \tan x(1 - \sec x) - (1 + \sec x)(-\sec x \tan x)}{(1 - \sec x)^2}$$
$$= \frac{\sec x \tan x - \sec^2 x \tan x + \sec x \tan x + \sec^2 x \tan x}{(1 - \sec x)^2}$$

and $\frac{dy}{dx} = \frac{2 \sec x \tan x}{(1 - \sec x)^2}$

It is NOT necessary to simplify this answer by attempting to use trigonometric identities! It is also standard procedure to leave the denominator in factored form.

Problem 6:

Differentiate $f(x) = x \ln x$

Here we must use the *Product Rule* with $u = x$ and $v = \ln x$.

Then $u' = 1$ and $v' = \frac{1}{x}$

Therefore, $f'(x) = \ln x + x(\frac{1}{x}) = \ln x + 1$

NOTE: Whenever possible you should ALWAYS combine like terms in your solution!

Problem 7:

Differentiate $f(x) = \frac{\ln x}{x}$

Here we must use the *Quotient Rule* with $u = \ln x$ and $v = x$.

Then $u' = \frac{1}{x}$ and $v' = 1$

Therefore, $f'(x) = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

NOTE: Whenever possible you should ALWAYS combine like terms in your solution!

Problem 8:

Differentiate $f(x) = x^2 e^x$

Here we must use the *Product Rule* with $u = x^2$ and $v = e^x$.

Then $u' = 2x$ and $v' = e^x$

Therefore, $f'(x) = 2xe^x + x^2 e^x$

Problem 9:

Differentiate $f(x) = e^x(\sin x + \cos x)$

Here we must use the *Product Rule* with $u = e^x$ and $v = \sin x + \cos x$.

Then $u' = e^x$ and $v' = \cos x - \sin x$

Therefore, $f'(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

Combining like terms, we get $f'(x) = 2e^x \cos x$

Problem 10:

Differentiate $f(x) = \frac{7x}{e^x}$

Here we must use the *Quotient Rule* with $u = 7x$ and $v = e^x$.

Then $u' = 7$ and $v' = e^x$

Therefore, $f'(x) = \frac{7e^x - 7xe^x}{e^{2x}} = \frac{7 - 7x}{e^x}$

NOTE: Whenever possible you should ALWAYS reduce your solutions to lowest terms!

Problem 11:

Differentiate $f(x) = \frac{2(3^x)}{x}$

Here we must use the *Quotient Rule* with $u = 2(3^x)$ and $v = x$.

Then $u' = 2\ln 3(3^x)$ and $v' = 1$

Therefore, $f'(x) = \dfrac{2\ln 3(3^x)x - 2(3^x)}{x^2}$

Please note that $2(3^x) \neq 6^x$.

Problem 12:

Differentiate $f(x) = \tan x \log_3 x$

Here we must use the *Product Rule* with $u = \tan x$ and $v = \log_3 x$.

Then $u' = \sec^2 x$ and $v' = \frac{1}{x\ln 3}$

Therefore, $f'(x) = \sec^2 x \log_3 x + \dfrac{\tan x}{x\ln 3}$

Problem 13:

Find the slope-intercept equation of the line tangent to the graph of $f(x) = 3x + \sin x$ at the point $(\pi, 3\pi)$.

Use the point-slope form $y - y_1 = m(x - x_1)$ with $m = f'(\pi)$.

Since $f'(x) = 3 + \cos x$ and $f'(\pi) = 3 + (-1) = 2$

then $y - 3\pi = 2(x - \pi)$

and $y = 2x + \pi$

Thus, slope-intercept equation of the line tangent to the graph of $f(x) = 3x + \sin x$ at the point $(\pi, 3\pi)$ is $y = 2x + \pi$.

Problem 14:

Determine ALL x-coordinates at which the graph of the function $f(x) = 2\cos x + x\sqrt{2}$ has a horizontal tangent line.

Since any horizontal line has a slope of $m = 0$, we must find the x-value at which $f'(x) = 0$. This is the x-coordinate of the point at which the slope of the tangent line is horizontal.

Since $f'(x) = -2\sin x + \sqrt{2}$, then

$$-2\sin x + \sqrt{2} = 0$$
$$\sin x = \frac{\sqrt{2}}{2}$$

Since we are asked to find ALL solutions, we ALWAYS find the solutions in the interval $[0, 2\pi)$ first.

NOTE: Since we are asked to find x-coordinates, we MUST find the solution(s) in radians and not degrees. However, it is easier to work with degrees! Therefore, if you wish, calculate the solution(s) in degrees first and then change them to radians.

$$x = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$
$$x = 45^\circ$$

We find the *Reference Angle* of angle $x = 45$ to be 45.

The numeric value of sine is positive for angles in QI and QII. Therefore, the solutions for x on the interval $[0, 2\pi)$ are

$$x_1 = 45^\circ \equiv \frac{\pi}{4}$$
$$x_2 = 180^\circ - 45^\circ = 135^\circ \equiv \frac{3\pi}{4}$$

We find that the tangent line is horizontal at ALL points with x-coordinates $\frac{\pi}{4} + 2\pi k$ and $\frac{3\pi}{4} + 2\pi k$, where k is any integer.