RELATIVE EXTREMA AND FIRST DERIVATIVE TEST

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Let's look at the graph of a quadratic function and some of its tangent lines.

As you can see, the tangent lines are increasing lines on the left side of the parabola and decreasing lines on the right side. From Algebra we know that increasing lines have a positive slope (m > 0) and the slope of decreasing lines is negative (m < 0).

Definition of Increasing/Decreasing Functions

A function is said to increase on an interval where $f'(x) > 0$. Likewise, a function is said to decrease on an interval where $f'(x) < 0$. When $f'(x) = 0$ there is neither an increase nor a decrease.

> NOTE: $f'(x) > 0$ actually means that the slope of the tangent line is positive and $f'(x) < 0$ means that the slope of the tangent lines is negative!

Absolute Extrema of a Function (singular: extremum)

The absolute extrema (absolute maximum or minimum) occur at the highest and lowest point, respectively in the domain of the function!

Relative Extrema of a Function

If a function has a relative extremum (relative maximum or minimum), then it will exist at a *critical number*. A relative extremum can be an absolute extremum if it is the highest or lowest point in the entire domain of the function.

Please note that the converse of the above statement is NOT true. That is, just because a function has *critical numbers* does not mean that relative extrema exist there.

In order to find relative extrema, we will use the following test.

First Derivative Test (FDT)

Let ***c*** be a *critical number* for a function ***f*** and let ***(a, b)*** denote an interval on the x-axis containing ***c***.

a. If ***f '(x)*** is positive for ***a < x < c*** and negative for ***c < x < b***, then ***f*** has a relative maximum at ***c***.
b. If ***f '(x)*** is negative for ***a < x < c*** and positive for ***c < x < b***, then ***f*** has a relative minimum at ***c***.
c. If ***f '(x)*** is positive or negative on both sides of ***c***, then ***f*** has no relative extrema at ***c***.

How to find the x-coordinates of *Relative Extrema (Maxima and Minima)*

1. Find the *critical number(s)* of the function ***f***.
2. Place the *critical number(s)* on a number line, thereby dividing the number line into two or more intervals.
3. Pick any test value ***t*** from each interval and find ***f '(t)***.

   a. If ***f '(t)*** is negative, place minus signs along the number line in the respective interval.
   b. If ***f '(t)*** is positive, place plus signs along the number line in the respective interval.

4. Use the *First Derivative Test* (FDT) to decide if the *critical numbers* are x-coordinates of *relative maxima* or *relative minima* or neither.

Relative extrema can exist at vertices which are parabolic in form or which are cusps. Cusps are points where two branches of a curve meet, end, and have the same tangent line.

We can use the following definition to evaluate cusps:

Let ***c*** be a *critical number.*

There is a cusp at the point $[c, f(c)]$ on the graph of the function ***f***, if either

- the graph of the derivative ***f '*** approaches negative infinity as ***x*** approaches ***c*** from the right AND positive infinity as ***x*** approaches ***c*** from the left

  OR

- the graph of the derivative ***f '*** approaches positive infinity as ***x*** approaches ***c*** from the right AND negative infinity as ***x*** approaches ***c*** from the left.

Problem 1:

Given $f(x) = 2x^3 + x^2 - 20x + 4$ with domain $(-\infty, \infty)$, find

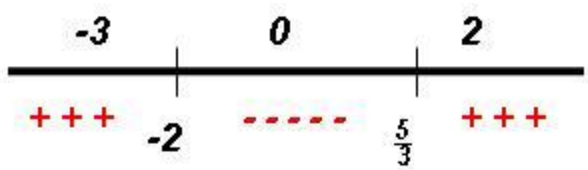
- the coordinates of any relative maximum and minimum points on the graph of the function

Step 1 - Find the *critical number(s)* of the function ***f***.

The first derivative is $f'(x) = 6x^2 + 2x - 20$.

The *critical numbers* are -2 and $\frac{5}{3}$.

Step 2 and Step 3 - Let's place the *critical numbers* on the number line and pick the test numbers -3, 0, and 2 from the respective intervals.



Then $f'(-3) = +28$,

$f'(0) = -20$,

and $f'(2) = +8$.

Please note the respective plus and minus signs in the intervals above!

Step 4 - By the FDT, the coordinates of the relative extrema are as follows:

relative maximum point $[-2, f(-2)] = (-2, 32)$

relative minimum point $[\frac{5}{3}, f(\frac{5}{3})] = (\frac{5}{3}, \frac{-467}{27})$

- the intervals over which the function is increasing and decreasing

Intervals of Increase are $(-\infty, -2)$ and $(\frac{5}{3}, \infty)$

Interval of Decrease is $(-2, \frac{5}{3})$

Problem 2:

Given $f(x) = x^{2/3} - 1$ with domain $(-\infty, \infty)$, find

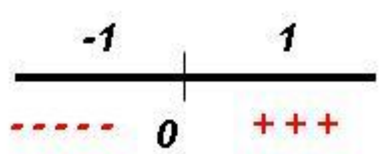
- the coordinates of any relative maximum and minimum points on the graph of the function

  Step 1 - Find the *critical number(s)* of the function $\boldsymbol{f}$.

  The first derivative is $f'(x) = \dfrac{2}{3x^{1/3}}$.

  The *critical number* is 0.

  Step 2 and Step 3 - Let's place the *critical number* on the number line and pick the test numbers -1 and 1 from the respective intervals.

  Then $f'(-1) = -\frac{2}{3}$ and $f'(1) = +\frac{2}{3}$.

  Step 4 - By the FDT, the coordinates of the relative extrema are as follows:

  NO relative maximum point

  relative minimum point $[0, f(0)] = (0, -1)$

  Since we are not dealing with a polynomial function, we must check to see if the relative minimum occurs at a vertex which is parabolic in form or which is a cusp.

  We find that $\lim_{x \to 0^+} \dfrac{2}{3x^{1/3}}$ approaches positive infinity and $\lim_{x \to 0^-} \dfrac{2}{3x^{1/3}}$ approaches negative infinity.

  By definition, the relative minimum occurs at a **cusp**.

- the intervals over which the function is increasing and decreasing

  Interval of Increase is $(0, \infty)$

  Interval of Decrease is $(-\infty, 0)$

Problem 3:

Given $f(x) = \sqrt[3]{x^2 - x - 2}$ with domain $(-\infty, \infty)$, find

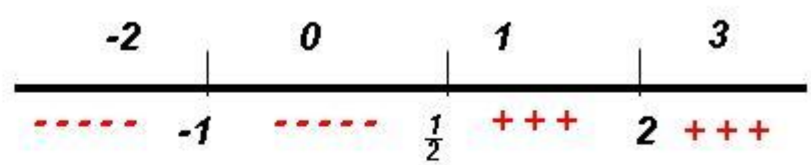
- the coordinates of any relative maximum and minimum points on the graph of the function

Step 1 - Find the *critical number(s)* of the function ***f***.

The first derivative is $f'(x) = \dfrac{2x - 1}{3(x^2 - x - 2)^{2/3}}$.

The *critical numbers* are -1, $\frac{1}{2}$, and 2.

Step 2 and Step 3 - Let's place the *critical numbers* on the number line and pick the test numbers -2, 0, 1, and 3 from the respective intervals.



Then $f'(-2) = \dfrac{-5}{+3(\sqrt[3]{16})}$,

$f'(0) = \dfrac{-1}{+3(\sqrt[3]{4})}$,

$f'(1) = \dfrac{+1}{+3(\sqrt[3]{4})}$,

and $f'(3) = \dfrac{+5}{+3(\sqrt[3]{16})}$.

Step 4 - By the FDT, the coordinates of the relative extrema are as follows:

NO relative maximum point

relative minimum point $[\frac{1}{2}, f(\frac{1}{2})] = (\frac{1}{2}, -\sqrt[3]{\frac{9}{4}})$

- the intervals over which the function is increasing and decreasing

Intervals of Increase are $(\frac{1}{2}, 2)$ and $(2, \infty)$

Intervals of Decrease are $(-\infty, -1)$ and $(-1, \frac{1}{2})$

Problem 4:

Given $f(x) = x\sqrt{9 - x^2}$ with domain $[-3,3]$, find

- the coordinates of any relative maximum and minimum points on the graph of the function

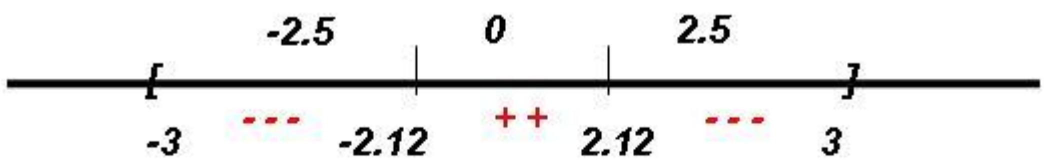
  Step 1 - Find the *critical number(s)* of the function ***f***.

  The first derivative is $f'(x) = \dfrac{9 - 2x^2}{(9 - x^2)^{1/2}}$.

  The *critical numbers* are -3, $-\frac{3\sqrt{2}}{2} \approx -2.12$, $\frac{3\sqrt{2}}{2} \approx 2.12$ and 3.

  Step 2 and Step 3 - Let's place the *critical numbers* on the number line and pick the test numbers -2.5, 0, and 2.5 from the respective intervals.

  Note that since -3 and 3 are endpoints of the domain we do not need to pick a test number to the left of -3 or to the right of 3! **This means that these two points could not be relative extrema!**



  Then $f'(-2.5) = \dfrac{-3.5}{+\sqrt{2.75}}$,

  $f'(0) = +3$,

  and $f'(2.5) = \dfrac{-3.5}{+\sqrt{2.75}}$.

  Step 4 - By the FDT, the coordinates of the relative extrema are as follows:

  relative maximum point $[\frac{3\sqrt{2}}{2}, f(\frac{3\sqrt{2}}{2})] = (\frac{3\sqrt{2}}{2}, \frac{9}{2})$

  relative minimum point $[-\frac{3\sqrt{2}}{2}, f(-\frac{3\sqrt{2}}{2})] = (-\frac{3\sqrt{2}}{2}, -\frac{9}{2})$

- the intervals over which the function is increasing and decreasing

  Interval of Increase is $(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$

  Intervals of Decrease are $(-3, -\frac{3\sqrt{2}}{2})$ and $(\frac{3\sqrt{2}}{2}, 3)$

Problem 5:

Given $f(x) = x\sqrt{2} - 2\cos x$ with restricted domain $[-2\pi, 2\pi]$, find

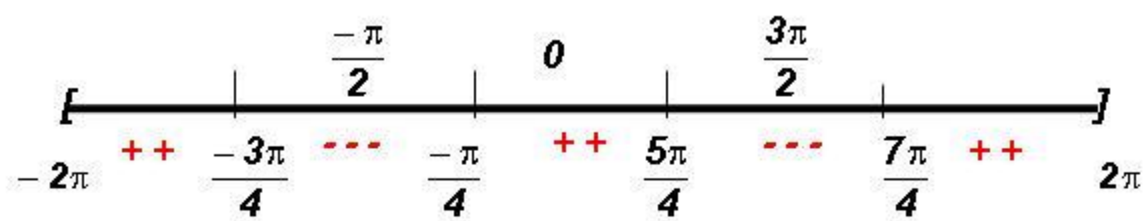
- the x-coordinates ONLY of any relative maximum and minimum points on the graph of the function

  Step 1 - Find the *critical number(s)* of the function ***f***.

  The first derivative is $f'(x) = \sqrt{2} + 2\sin x$.

  The *critical numbers* are $\frac{-3\pi}{4}$, $\frac{-\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

  Step 2 and Step 3 - Let's place the *critical numbers* on the number line and pick the test numbers -2π, $-\frac{\pi}{2}$, 0, $\frac{3\pi}{2}$ and 2π from the respective intervals.



  Then $f'(-2\pi) = +\sqrt{2}$,

  $f'\left(\frac{-\pi}{2}\right) = \sqrt{2} - 2 \approx -0.59$,

  $f'(0) = +\sqrt{2}$,

  $f'\left(\frac{3\pi}{2}\right) = \sqrt{2} - 2 \approx -0.59$,

  and $f'(2\pi) = +\sqrt{2}$.

  Step 4 - By the FDT, the x- coordinates of the relative extrema are as follows:

  x- coordinates of relative maximum point $\frac{-3\pi}{4}$ and $\frac{5\pi}{4}$

  x- coordinates of relative minimum point $\frac{-\pi}{4}$ and $\frac{7\pi}{4}$

- the intervals over which the function is increasing and decreasing

Intervals of Increase are $\left[-2\pi, \frac{-3\pi}{4}\right)$, $\left(\frac{-\pi}{4}, \frac{5\pi}{4}\right)$, and $\left(\frac{7\pi}{4}, 2\pi\right]$

Intervals of Decrease are $\left(\frac{-3\pi}{4}, \frac{-\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Problem 6:

Given $f(x) = (x+2)^3 - 4$ with domain $(-\infty, \infty)$, find

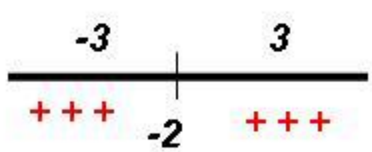
- the coordinates of any relative maximum and minimum points on the graph of the function

Step 1 - Find the *critical number(s)* of the function ***f***.

The first derivative is $f'(x) = 3(x+2)^2$.

The *critical number* is -2.

Step 2 and Step 3 - Let's place the *critical number* on the number line and pick the test numbers -3 and 3 from the respective intervals.



Then $f'(-3) = 3$ and $f'(3) = 12$.

Step 4 - By the FDT, there are NO relative maximum and minimum points.

- the intervals over which the function is increasing and decreasing

Intervals of Increase are $(-\infty, -2)$ and $(-2, \infty)$.

No Intervals of Decrease