$$\lim_{x \to \infty} \int_{2}^{3} \frac{1}{dx} dy$$

RELATED RATES

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In all of the natural sciences and many of the social and behavioral sciences, quantities are encountered that are **related**, but which **vary with time**. Problems involving rates of related quantities are referred to as *related rate problems*.

In such problems we want to find the rate at which one of the related quantities is changing with respect to time, while the rates at which the other quantities are changing with respect to time are known.

Helpful Guidelines for Solving *Related Rate Problems*:

1. Identify given rates of change as derivatives.

   For example:

   "4 cubic meters per second" is a rate of change (meters per second). It relates to volume V (cubic meters) and this rate of change would be expressed as

   $$\frac{dV}{dt} = 4$$

2. Identify given quantities and assign variables to them. If possible, draw a picture!

   For example:

   H is the height, L is the length, t is the time in seconds

3. State what rate has to be found under what condition.

   For example:

   Find $\frac{dH}{dt}$ **when** $H = 3$

4. Taking into account (a) the given rate of change, (b) the given quantities, and (c) the rate of change that is to be found, write a **primary** equation that expresses the relationship among them. Be careful, sometimes you will find more variables than needed to write an appropriate primary equation!

5. If necessary, reduce the **primary** equation to an equation in two variables. This may involve the use of one or more **secondary** equations relating the variables of the primary equation.

NOTE: DO NOT yet use the value *H = 3* given as a condition for finding the rate. You cannot use this value until Step 7 otherwise you would not be able to calculate the rate!!!

6. Differentiate the primary equation found in Step 5 with respect to time ***t***. Then solve for the rate you were asked to find.

7. Substitute the value given as a condition for finding the rate NOW. Express your answer in a sentence.

Problem 1:

Air is being pumped into a spherical balloon at a rate of 4 cubic inches per minute. Find the rate of change of the radius when the radius is 3 inches. Round your answer to 3 decimal places.

1. The given rate of change is

$$\frac{dV}{dt} = 4$$

2. The quantities involved are

r is the radius of the balloon

3. We wish to find $\frac{dr}{dt}$ when $r = 3$.

4. The relationship among the quantities can be expressed by the primary equation $V = \frac{4}{3}\pi r^3$, which is the volume of a sphere.

5. The primary equation is already an equation in two variables.

NOTE: We are not allowed to use the value *r = 3* until Step 7 because it is the condition for finding the rate *dr/dt* !!!

6. Let's differentiate both sides of $V = \frac{4}{3}\pi r^3$ with respect to ***t***.

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

and $$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

7. Given $\frac{dV}{dt} = 4$ and $r = 3$, we find

$$\frac{dr}{dt} = \frac{1}{4\pi(3)^2}(4) = \frac{1}{9\pi}$$

The rate of change of the radius of the balloon is approximately ***0.035* inches per minute** when $r = 3$.

Problem 2:

A pebble is dropped into a calm pond causing ripples in the form of concentric circles. The radius of the outermost ripple is increasing at a constant rate of 2 feet per second. When the radius is 6 feet, at what rate is the total area of the disturbed water changing? Round your answer to 3 decimal places.

1. The given rate of change is

$$\frac{dr}{dt} = 2$$

2. The quantities involved are

   r is the radius of the outermost ripple

3. We wish to find $\frac{dA}{dt}$ when $r = 6$.

4. The relationship among the quantities can be expressed by the primary equation $A = \pi r^2$, which is the area of a circle.

5. The primary equation is already an equation in two variables.

   NOTE: We are not allowed to use the value *r = 6* until Step 7 because it is the condition for finding the rate *dA/dt* !!!

6. Let's differentiate both sides of $A = \pi r^2$ with respect to ***t***.

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$
$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right)$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

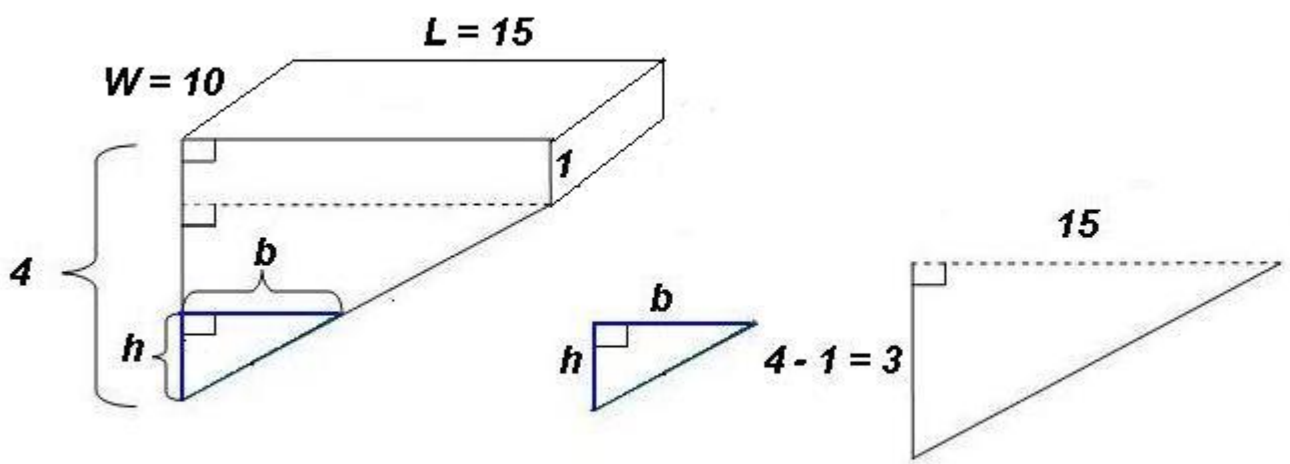
7. Given $\frac{dr}{dt} = 2$ and $r = 6$, we find

$$\frac{dA}{dt} = 2\pi(6)(2) = 24\pi$$

The rate of change of the area of the disturbed water is approximately ***75.398*** **square feet per second** when $r = 6$.

Problem 3:

A rectangular swimming pool is 10 meters wide and 15 meters long. It is 4 meters deep at one end and 1 meter deep at the other end. **See picture below!** If water is pumped into the pool at the rate of 0.4 cubic meters per minute, find the rate of change of the water level when the water is 2 meters high at the deep end of the pool. Round your answer to 3 decimal places.



1. The given rate of change is

$$\frac{dV}{dt} = 0.4$$

2. The quantities involved are

L is the length of the pool

W is the width of the pool

D_1 and D_2 are the two depths of the pool, with D_1 being the deeper of the two

h is the height of the water level at which the rate of change is supposed to be found

b is the width of the water surface when the water level equals ***h***

3. We wish to find $\frac{dh}{dt}$ when $h = 2$.

4. In this problem, we will only consider the *triangular solid* with width b, height h, and length $W = 10$. The relationship among these quantities can then be expressed by the primary equation $V = WA$, where W is the width of the pool (but the length of the *triangular solid*) and A is the area of the triangle with base b and height h.

NOTE: When calculating rates of change as something is poured into a container or drained out of a container, we only use THAT portion of the container at which the condition for finding the requested rate is met.

5. Since the primary equation is an equation in several variables, we must find secondary equations so that we can write the primary equation in terms of two variables.

Let's use the fact that the area of a triangle is $A = \frac{1}{2}bh$, where b is called the base and h is the vertical distance from the base to its opposite angle. We also know that $W = 10$.

Now we can write the volume as $V = 10(\frac{1}{2}bh) = 5bh$. Since we still have an equation in three variables we must find another way to reduce it to two variables.

Whenever you can, try to use the concept of *Similar Triangles*.

If two triangles are similar, then one is an enlargement of the other. This means that the two triangles will have the same angles and their sides will be in the same proportion (e.g. the sides of one triangle will all be 3 times the sides of the other, etc.).

In this problem, we will use the two similar triangles shown in the picture above.

Using $L = 15$ and $D_1 - D_2 = 4 - 1 = 3$, we can write

$\frac{b}{15} = \frac{h}{3}$ and we will solve this for b!

Note we cannot solve for h because then V would not contain h anymore, but only b. This would not work since we have to find dh/dt.

$b = 5h$

NOTE: We are not allowed to use the value $h = 2$ until Step 7 because it is the condition for finding the rate dh/dt !!!

Subsequently, $V = 5(5h)h = 25h^2$.

6. Let's differentiate both sides of $V = 25h^2$ with respect to $\boldsymbol{t}$.

$$\frac{d}{dt}(V) = \frac{d}{dt}(25h^2)$$
$$\frac{dV}{dt} = 50h\frac{dh}{dt}$$

and $\frac{dh}{dt} = \frac{1}{50h}\frac{dV}{dt}$

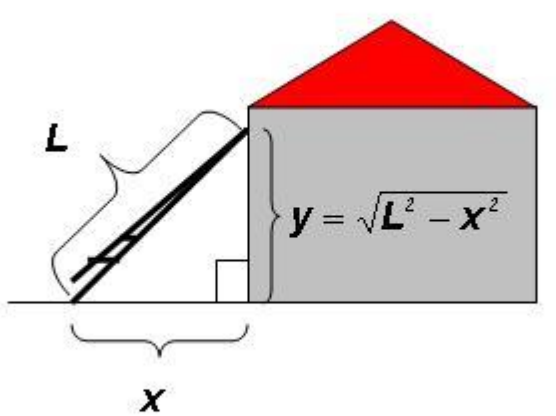
7. Given $\frac{dV}{dt} = 0.4$ and $h = 2$, we find

$$\frac{dh}{dt} = \frac{1}{50(2)}(0.4) = \frac{1}{250}$$

The rate of change of the water level is exactly ***0.004*** **meters per minute** when $h = 2$.

Problem 4:

A ladder 25 meters long leans against a house. **See picture below!** Find the rate at which the top of the ladder is moving toward the ground when the foot of the ladder is 15 meters away from the house and sliding along the ground away from the house at the rate of 2 meters per second.



1. The given rate of change is

$$\frac{dx}{dt} = 2$$

2. The quantities involved are

***L* = 25** is the length of the ladder

y is the vertical distance from the top of the ladder to the ground

x is the distance from the foot of the ladder to the house

3. We wish to find $\frac{dy}{dt}$ when $x = 15$.

4. The relationship among the quantities can be expressed by the *Pythagorean Theorem* since the ladder makes a right triangle with the house. Therefore, the primary equation is
$x^2 + y^2 = L^2$

5. Since we are dealing with an equation in three variables, we must find secondary equations so that we can write the primary equation in terms of two variables.

Let's use the fact that $L = 25$ and so that $x^2 + y^2 = 25^2$.

NOTE: We are not allowed to use the value *x = 15* nor are we allowed to use the fact that $y = \sqrt{625 - x^2}$ until Step 7 because they are the condition for finding the rate *dy/dt* !!!

6. Let's differentiate both sides of $x^2 + y^2 = 625$ with respect to ***t***.

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(625)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and $$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = \frac{-x}{y}\frac{dx}{dt}$$

7. Given $\frac{dx}{dt} = 2$, $x = 15$, and $y = \sqrt{625 - x^2}$, we get

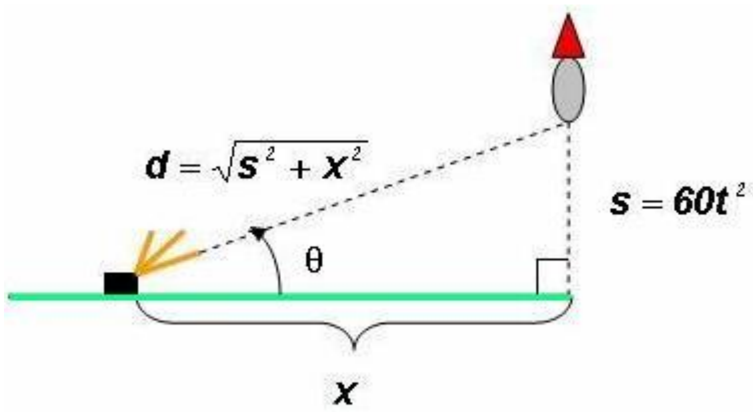
$$\frac{dy}{dt} = \frac{-x}{\sqrt{625 - x^2}}\frac{dx}{dt} = \frac{-15}{\sqrt{625 - 225}}(2) = \frac{-30}{20}$$

and $\frac{dy}{dt} = -\frac{3}{2}$ Please note that the minus sign indicates that the ladder is sliding "downward."

The rate of change at which the top of the ladder is moving toward the ground is exactly ***1.5* meters per second** when $x = 15$.

Problem 5:

A television camera at ground level is filming the lift-off of a space shuttle that is rising vertically from the ground according to the position equation $s = 60t^2$, where s is measured in feet and t is measured in seconds. The camera is 2,400 feet from the launch pad. **See picture below!** Find the rate of change in the angle of elevation of the camera 10 seconds after lift-off. (Assume the Earth is flat for the purposes of this exercise.) Round your answer to 3 decimal places.



1. There is no rate of change given.

2. The quantities involved are

   s is the vertical distance from the launch pad to the shuttle

   x is the horizontal distance from the launch pad to the camera

   d is the distance from the camera to the shuttle

   θ is the *Angle of Elevation* of the camera

   t is the time in seconds

3. We wish to find $\dfrac{d\theta}{dt}$ when $t = 10$.

   4. The relationship among the quantities can be expressed by the primary equation $\tan\theta = \dfrac{s}{x}$, since we are dealing with a right triangle.

   Note: We could have also used $\cos\theta = \dfrac{x}{d}$ **or** $\sin\theta = \dfrac{s}{d}$**.**

5. Since the primary equation contains three variables, we must find secondary equations so that we can write it n terms of two variables.

Let's use the facts that $x = 2400$ and $s = 60t^2$ so that we can write

$$\tan\theta = \frac{60t^2}{2400} = \frac{1}{40}t^2$$

NOTE: We are not allowed to use the value *t = 10* until Step 7 because it is the condition for finding the rate *dθ/dt* !!!

6. Now let's differentiate both sides of $\tan\theta = \frac{1}{40}t^2$ with respect to ***t***.

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{1}{40}t^2\right)$$
$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{40}(2t)$$
$$\frac{d\theta}{dt} = \frac{t}{20\sec^2\theta} = \frac{t}{20}\cos^2\theta$$

7. We know that $\cos\theta = \frac{x}{d}$, $x = 2400$, $d = \sqrt{s^2 + x^2}$, $s = 60t^2$, and $t = 10$.

Therefore, we can express ***cos*** **Θ** as follows:

$$\cos\theta = \frac{x}{d}$$
$$= \frac{x}{\sqrt{s^2 + x^2}}$$
$$= \frac{2400}{\sqrt{(60t^2)^2 + 2400^2}}$$
$$= \frac{2400}{\sqrt{3600t^4 + 2400^2}}$$
$$= \frac{2400}{\sqrt{3600(10)^4 + 2400^2}}$$

Then the derivative becomes

$$\frac{d\theta}{dt} = \frac{10}{20}\left(\frac{2400}{\sqrt{3600(10)^4 + 2400^2}}\right)^2$$

and $$\frac{d\theta}{dt} = \frac{1}{2}\left(\frac{2400^2}{3600(10000) + 2400^2}\right)$$

Finally, $\frac{d\theta}{dt} = \frac{2}{29}$

The rate of change of the angle of elevation is approximately ***0.069*** **radians per second** at $t = 10$.