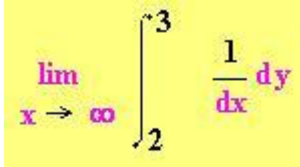

OPTIMIZATION

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Second Derivative Test (SDT)

Let c be a *critical number* for a function f .

a. If $f''(c) > 0$ (positive), then f has a relative minimum at c.
b. If $f''(c) < 0$ (negative), then f has a relative maximum at c.
c. If $f''(c) = 0$ use the *First Derivative Test*.

Formulas for Business and Science Applications

Position Function: $s(t)$

The position of a moving object at time t.

Instantaneous Velocity or Velocity: $v(t) = s'(t)$

The rate of change of an object's position with respect to time t.

Instantaneous Acceleration or Acceleration: $a(t) = v'(t) = s''(t)$

The rate of change of an object's velocity with respect to time t.

Cost Function: $C(x)$

The cost of producing/selling x items.

Demand (Price) Function: $p(x)$

The selling price of an item expressed as a function of x items produced/sold.

Revenue Function: $R(x) = x\,p(x)$ or $R(x) = xp$, where $p(x)$ is the demand function and p a fixed price.

The amount of money received from selling x items.

Profit Function: $P(x) = R(x) - C(x)$

The amount of money earned from selling x items.

Review of Useful Geometry Formulas:

Rectangle with Length x and Width y :

Area: $A = xy$

Perimeter: $P = 2x + 2y$

Rectangular Solid with Length x, Width y, and Height z :

Volume: $V = xyz$

Surface Area: $S = 2xy + 2yz + 2xz$ (area of top and bottom plus area of front and back plus area of right and left side)

Circle with Radius r :

Area: $A = \pi r^2$

Circumference: $C = 2\pi r$

Cylinder with Radius r and Height h :

Volume: $V = \pi r^2 h$

Surface Area: $S = 2\pi r^2 + 2\pi rh$ (area of circular top and bottom plus area of rectangle wrapped around the perimeter of the top and bottom)

Triangle

Area: $A = \frac{1}{2} bh$, where b is called the base and h is the vertical distance from the base to its opposite angle.

Problem 1:

A projectile is fired directly upward with its height (in feet) above the ground after t seconds given by $s(t) = 192t - 16t^2$. Find the following:

a. the velocity after t seconds

$$v(t) = s'(t) = 192 - 32t$$

b. the acceleration after t seconds

$$a(t) = v'(t) = -32$$

c. the maximum height (use the Second Derivative Test to prove it!)

We must look for a relative maximum by differentiating ***s(t)*** to find critical numbers. Remember that relative extrema exist at critical numbers!!!

$$s'(t) = 192 - 32t$$

$$192 - 32t = 0$$

$t = 6$, which is a critical number.

Using the SDT, we find that $s''(t) = -32$ and $s''(6) = -32 < 0$, which indicates that there is a relative maximum at $t = 6$.

Therefore, the maximum height is s(6) = 576 ft.

Problem 2:

Given is a demand (price) function $p(x) = 80 - 0.2x$ in dollars and a cost function $C(x) = 5x + 10$, where x is the number of items produced. Find

a. the revenue function

$$R(x) = xp(x) = 80x - 0.2x^2$$

b. the profit function

$$P(x) = R(x) - C(x) = -0.2x^2 + 75x - 10$$

c. the maximum profit (use the Second Derivative Test to prove it!)

We must look for a relative maximum by differentiating ***P(x)*** to find critical numbers. Remember that relative extrema exist at critical numbers!!!

$$P'(x) = 80 - 0.4x$$

$$80 - 0.4x = 0$$

$x = 187.5$, which is a critical number.

Using the SDT, we find that $P''(x) = -0.4$ and $P''(187.5) = -0.4 < 0$, which indicates that there is a relative maximum at $x = 187.5$.

Therefore, the maximum profit is P(187.5) = 7,021.25 dollars.

Problem 3:

A manufacturer determines that x units of a product will be sold if the selling price is $p(x) = 400 - 0.05x$ for each unit. If the production cost for x units is $C(X) = 500 + 10x$, find

a. the revenue function

$$R(x) = xp(x) = 400x - 0.05x^2$$

b. the profit function

$$P(x) = R(x) - C(x) = -0.05x^2 + 390x - 500$$

c. the maximum profit (use the Second Derivative Test to prove it!)

We must look for a relative maximum by differentiating $P(x)$ to find critical numbers. Remember that relative extrema exist at critical numbers!!!

$P'(x) = -0.1x + 390$

$-0.1x + 390 = 0$

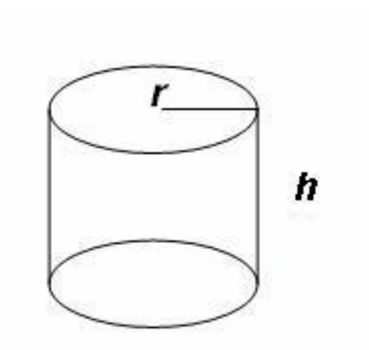
$x = 3900$, which is a critical number.

Using the SDT, we find that $P''(x) = -0.1$ and $P''(3900) = -0.1 < 0$, which indicates that there is a relative maximum at $x = 3900$.

Therefore, the maximum profit is P(3900) = 760,000 dollars.

Problem 4:

A metal cylindrical container with an open top is to hold 1 cubic foot. If there is no waste in construction, find the dimensions that require the least amount of material. Round your answers to two decimal places.



1. The quantities involved are

r is the radius of the cylinder

h is the height of the cylinder

V is the volume of the cylinder

2. We must find the radius r and the height h that minimize the surface area S.

3. The **primary equation** is $S = 2\pi rh + \pi r^2$.

4. Let's reduce the primary equation to an equation in two variables.

   Given $V = 1$, we can use the secondary equation $V = \pi r^2 h$ to write $1 = \pi r^2 h$, which we can solve for either r or h.

   Since it is easier to solve for h, we can then find $h = \frac{1}{\pi r^2}$.

   Finally, we can change $S = 2\pi rh + \pi r^2$

   to $S(r) = 2\pi r\left(\frac{1}{\pi r^2}\right) + \pi r^2 = 2r^{-1} + \pi r^2$.

5. Let's find the relative extrema of $S(r) = 2r^{-1} + \pi r^2$.

   We find the derivative to be $S'(r) = -2r^{-2} + 2\pi r = \frac{-2 + 2\pi r^3}{r^2}$.

   Finding *critical numbers* when $S'(r) = 0$:

$$0 = \frac{-2 + 2\pi r^3}{r^2}$$
$$0 = -2 + 2\pi r^3$$
$$\pi r^3 = 1$$
$$r^3 = \frac{1}{\pi}$$

   and $r = \sqrt[3]{\frac{1}{\pi}} \approx 0.68$

   Finding *critical numbers* when $S'(r)$ is undefined:

$$r^2 = 0$$

   $r = 0$ is not a critical number since the radius cannot be equal to 0.

6. Now we'll use the *Second Derivative Test* to see if a relative minimum exists at $r = \sqrt[3]{\frac{1}{\pi}} \approx 0.68$.

$$S''(r) = 4r^{-3} + 2\pi = \frac{4}{r^3} + 2\pi$$

$S''\left(\frac{1}{\sqrt[3]{\pi}}\right) = 6\pi > 0$, which indicates that there is indeed a relative minimum at $r = \sqrt[3]{\frac{1}{\pi}} \approx 0.68$.

7. Finally, we find the dimensions that minimize the surface area.

Since $r = \sqrt[3]{\frac{1}{\pi}} \approx 0.68$ minimizes the surface area, we can now find the height h that minimizes the surface area by using the volume formula $1 = \pi r^2 h$.

That is,

$$1 = \pi\left(\sqrt[3]{\frac{1}{\pi}}\right)^2 h$$

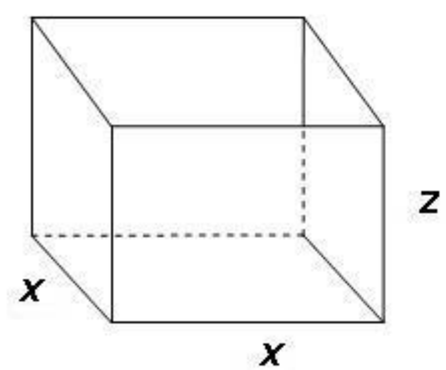
$$1 = \frac{\pi}{\pi^{2/3}} h$$

$$h = \sqrt[3]{\frac{1}{\pi}}$$

The surface area is minimized given an approximate height and approximate radius of 0.68 ft each.

Problem 5:

If a box with a square base and open top is to have a volume of 4 cubic feet, find the dimensions that require the least amount of material.



NOTE:
The box has a
square base!

1. The quantities involved are

   x is the width and length of the box

   z is the height of the box

   V is the volume of the box

2. We must find the width x, the length x, and the height z that minimize the surface area S.

3. The **primary equation** is $S = x^2 + 4xz$.

4. Let's reduce the primary equation to an equation in two variables.

   Given $V = 4$, we can use the secondary $V = x^2 z$ to write $4 = x^2 z$, which we can then solve for either x or z.

   Since it is easier to solve for z, we find $z = \frac{4}{x^2}$.

   Finally, we can change $S = x^2 + 4xz$

   to $S(x) = x^2 + 4x\left(\frac{4}{x^2}\right) = x^2 + 16x^{-1}$.

5. Let's find the relative extrema of $S(x) = x^2 + 16x^{-1}$.

   We find the derivative to be $S'(x) = 2x - 16x^{-2} = \frac{2x^3 - 16}{x^2}$.

   Finding *critical numbers* when $S'(x) = 0$:

   $$0 = \frac{2x^3 - 16}{x^2}$$
   $$0 = 2x^3 - 16$$
   $$x^3 = 8$$

   and $x = 2$

   Finding *critical numbers* when $S'(x)$ is undefined:

   $$x^2 = 0$$
   $x = 0$ is not a critical number since the width/length cannot be equal to 0

6. Now we'll use the *Second Derivative Test* to see if a relative minimum exists at $x = 2$.

$$S''(x) = 2 + 32x^{-3} = 2 + \frac{32}{x^3}$$

$S''(2) = 6 > 0$, which indicates that there is indeed a relative minimum at $x = 2$.

7. Since $x = 2$ minimizes the surface area we can now find the height z that minimizes the surface area by using the volume formula $4 = x^2 z$.

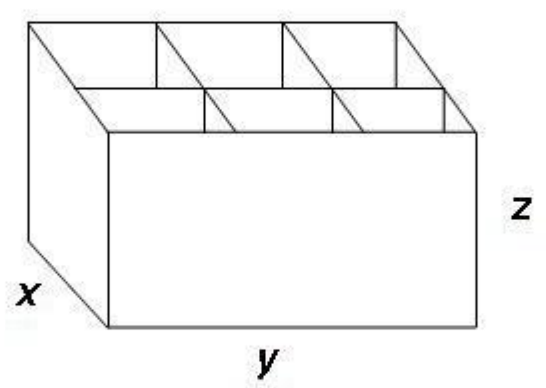
   That is,

$$4 = (2)^2 z$$
$$z = 1$$

The dimensions that minimize the surface area of the box are a width and length of 2 ft and a height of 1 ft.

Problem 6:

1,000 feet of chain link fence will be used to construct 6 rectangular animal cages side-by-side arranged in two rows and three columns. Find the dimensions that maximize the enclosed area.



1. The quantities involved are

   x is the width of the total cage area

   y is the length of the total cage area

   z is the height of the cages

   P is the perimeter of the total cage area

2. We must find the width x and the length y that maximize the floor area A of all cages.

3. The **primary equation** is $A = xy$.

4. Let's reduce the primary equation to an equation in two variables.

   Given $P = 1000$, we can use the secondary equation $P = 4x + 3y$ to write $1000 = 4x + 3y$, which we can then solve for either x or y.

   Let's solve for x, to find $x = \dfrac{1000 - 3y}{4}$.

   Finally, we can change $A = xy$

   to $A(y) = \left(\dfrac{1000 - 3y}{4}\right)y = 250y - \frac{3}{4}y^2$.

5. Let's find the relative extrema of $A(y) = 250y - \frac{3}{4}y^2$.

   We find the derivative to be $A'(y) = 250 - \frac{3}{2}y$.

   Finding *critical numbers* when $A'(y) = 0$:

   $$0 = 250 - \tfrac{3}{2}y$$

   and $y = \dfrac{500}{3}$

   Since the derivative does not have a denominator containing a variable, no other *critical numbers* can be found.

6. Now we'll use the *Second Derivative Test* to see if a relative maximum exists at $y = \dfrac{500}{3}$.

   $$A''(y) = -\tfrac{3}{2}$$

   $A''\left(\dfrac{500}{3}\right) = -\dfrac{3}{2} < 0$, which indicates that there is indeed a relative maximum at $y = \dfrac{500}{3}$.

7. Since $\frac{500}{3}$ maximizes the length ***y*** of the total cage area, we can now find the width ***x*** that maximizes the total cage area by using the perimeter formula $1000 = 4x + 3y$.

   That is,

   $$1000 = 4x + 3\left(\frac{500}{3}\right)$$

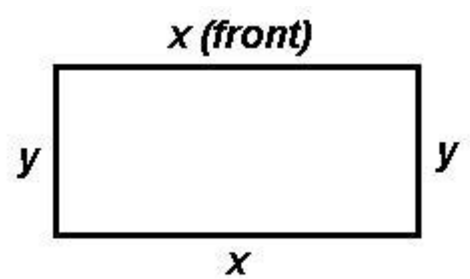
   $$500 = 4x$$

   and $x = 125$

   The floor area for all cages is maximized given a width of 125 ft and a length of 166 $\frac{2}{3}$ ft.

Problem 7:

A builder wishes to fence in 50,000 square meters of land on his property in a rectangular shape. Because of security reasons, the fence along the front part of the land will cost \$3 per meter, while the fence for the other three sides will cost \$2 per meter. How much of each type of fence will the builder have to buy in order to minimize the cost of the fence? What is the minimum cost?



1. The quantities involved are

   x is the length of the land to be fenced in

   y is the width of the land to be fenced in

   C is the cost of the fence

   A is the area of the land to be fenced in

2. We must find the length ***x*** and the width ***y*** that minimizes the cost ***C*** of the fence.

3. The **primary equation** is the cost function $C = 3x + 2x + 2y + 2y = 5x + 4y$. Please note that we multiplied the lengths of the sides by their respective dollar amount per meter!

4. Let's reduce the primary equation to an equation in two variables.

   Given $A = 50000$, we can use the secondary equation $A = xy$ to write $50000 = xy$, which we can then solve for either ***x*** or ***y***.

Let's solve for ***y***, to find $y = \frac{50000}{x}$.

Finally, we can change ***C = 5x + 4y***

to $C = 5x + 4\left(\frac{50000}{x}\right) = 5x + \frac{200000}{x}$.

5. Let's find the relative extrema of $C(x) = 5x + \frac{200000}{x}$.

We find the derivative to be $C'(x) = 5 - \frac{200000}{x^2}$.

Finding *critical numbers* when ***C'(x) = 0***:

$$0 = 5 - \frac{200000}{x^2}$$
$$-5 = -\frac{200000}{x^2}$$
$$-5x^2 = -200000$$
$$x^2 = 40000$$
$$x = \pm 200$$

Since we are working with measurements, we can rule out ***-200*** immediately since the length cannot be negative!

Finding *critical numbers* when ***C'(x)*** is undefined:

$$x^2 = 0$$
$x = 0$ is not a critical number since the width/length cannot be equal to ***0***

6. Now we'll use the *Second Derivative Test* to see if a relative minimum exists at ***x = 200***.

$$C''(x) = \frac{400000}{x^3}$$

$C''(200) = \frac{400000}{(200)^3} = 0.05 > 0$, which indicates that there is a relative minimum at ***x = 200***.

7. Since a length of 200 meters minimizes the cost of the fence, we can now find the width y that minimizes the cost of the fence by using the area formula $50000 = xy$.

   That is,

   $$50000 = 200y$$

   $$y = 250$$

   The builder will have to buy 200 m of fence costing \$3 per meter and 700 m of fence costing \$2 per meter.

   Given the cost function $C = 5x + 4y$ we find the cost of the fence to be \$2000.

Problem 8:

An orchard has an average yield of 25 bushels per tree when there are at most 40 trees per acre. When there are more than 40 trees per acre, the average yield decreases by 1/2 bushel per tree for every tree over 40. Find the number of trees per acre that will give the greatest yield per acre.

1. The quantity involved is

   x is the number of trees per acre

2. We must find how many trees per acre will maximize the yield per acre.

3. The **primary equation** is a piecewise-defined function:

   a. When at most 40 trees per acre are planted, the yield function becomes $f(x) = 25x$.

   b. When we plant more than 40 trees per acre, the number of bushels produced by each tree becomes $25 - \frac{1}{2}(x - 40)$.

      Here $x - 40$ is the number of trees over 40, and $\frac{1}{2}(x - 40)$ is the corresponding decrease in bushels per tree.

      Then, when more than 40 trees per acre are planted, the yield function becomes $f(x) = [25 - \frac{1}{2}(x - 40)]x = 45x - \frac{1}{2}x^2$.

   Therefore, the **primary equation** is

   $$f(x) = \begin{cases} 25x & \text{if } 0 < x \leq 40 \\ 45x - \frac{1}{2}x^2 & \text{if } x > 40 \end{cases}$$

4. This primary equation is already reduced to two variables.

5. Let's find the relative extrema of the primary equation.

   a. For $f(x) = 25x$, we find the first derivative to be $f'(x) = 25$. Therefore, there are no critical numbers between 0 and 40.

      However, we do have a maximum. In this case there is an absolute maximum at ***x = 40***.

      That is, if we plant 40 trees per acre we will have a yield of $f(40) = 25(40)$ or 1000 bushels.

   b. For $f(x) = 45x - \frac{1}{2}x^2$, we find the first derivative to be $f'(x) = 45 - x$ and

      $$0 = 45 - x$$
      $$x = 45$$
      , which is a critical number.

6. Now we'll use the *Second Derivative Test* to see if a relative maximum exists at ***x = 45***.

   $f''(x) = -1$

   $f''(45) = -1 < 0$, which indicates that there is indeed a relative maximum at ***x = 45***.

   That is, if we plant 45 trees per acre will have a maximum yield of $f(45) = 45(45) - \frac{1}{2}(45)^2$ or 1012.5 bushels. That's a bigger yield than given 40 trees.

7. **Therefore, we must plant 45 trees per acre to achieve the greatest yield per acre.**