$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx}\, dy$$

INTRODUCTION TO LIMITS

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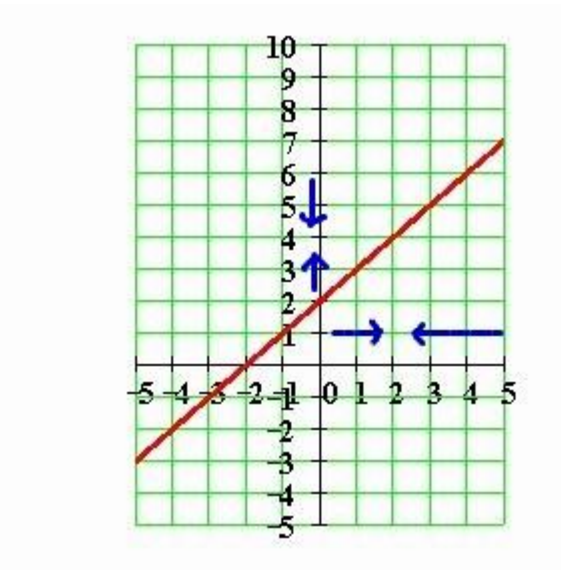
Warm-up questions:

Given $g(x) = x + 2$, assume that we are approaching ***2*** along the x-axis from the right and from the left. At the same time, what value do we approach along the y-axis?

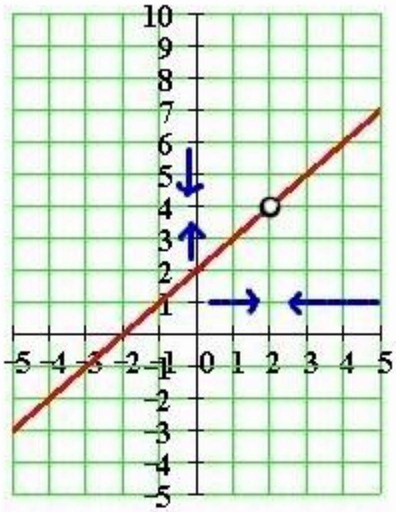
NOTE: "approach" DOES NOT mean we have "arrived" at *2* !!!

From the picture below, we can see that we are approaching the y-value of ***4*** as we approach the value of ***2*** along the x-axis.



Given $f(x) = \dfrac{x^2 - 4}{x - 2} = \dfrac{(x-2)(x+2)}{x-2}$, assume that we are approaching ***2*** along the x-axis from the right and from the left. At the same time, what value do we approach along the y-axis?

From the picture below, we can see that the graphs of ***f*** and ***g*** are almost identical except that the graph of ***f*** has a hole at ***(2, 4)***. However, we can also see that we are approaching the y-value of ***4*** as we approach the value of ***2*** along the x-axis.



Informal Definition of the Limit using Graphs of Functions:

If we can answer "YES" to the question

"Given the graph of any function ***f***, do we approach a **SPECIFIC REAL** number ***L*** on the y-axis as we approach a **REAL** number ***c*** on the x-axis from the right and from the left?",

then we can write $\lim_{x \to c} f(x) = L$.

NOTE:

$\lim_{x \to c} f(x) = L$ is pronounced as the *limit of* ***f(x)*** *as* ***x*** *approaches* ***c*** *equals* ***L***.

Now we can reformulate our two warm-up questions to include their answers:

- Question 1 becomes $\lim_{x \to 2} (x + 2) = 4$
- Question 2 becomes $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$

Please note that $g(2) = 2 + 2 = 4$ is defined, but $f(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$ is undefined!!! Finding the answer to ***g(2)*** and ***f(2)*** means that we want to find the y-value after having "arrived" at ***2*** on the x-axis

Again, be very careful NOT TO CONFUSE "approach" with "arrive."

One-Sided Limits

$\lim_{x \to c^+} f(x)$ indicates the right-sided limit (see c^+), that is, the limit of ***f(x)*** as we approach ***c*** from the **right** only.

$\lim_{x \to c^-} f(x)$ indicates the left-sided limit (see c^-), that is, the limit of ***f(x)*** as we approach ***c*** from the **left** only.

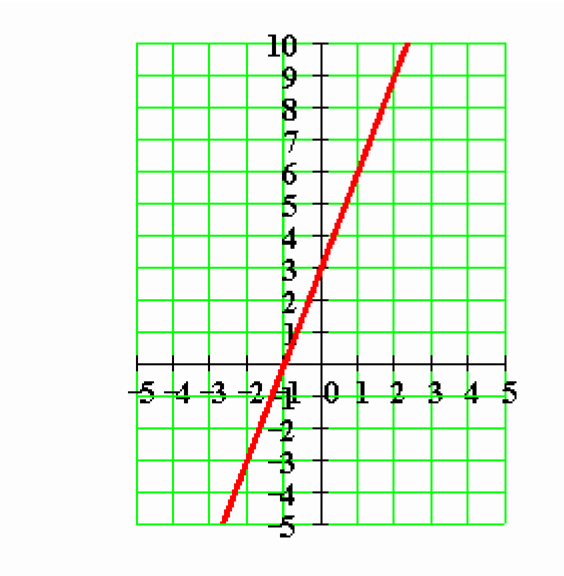
Relationship between THE Limit and the One-Sided Limits:

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$

In Problems 1-12, note that $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ exist, $f(c)$ is defined, and they all have the same value! This usually happens when "c" is in the domain of a function, BUT NOT an endpoint of the domain!

Problem 1:

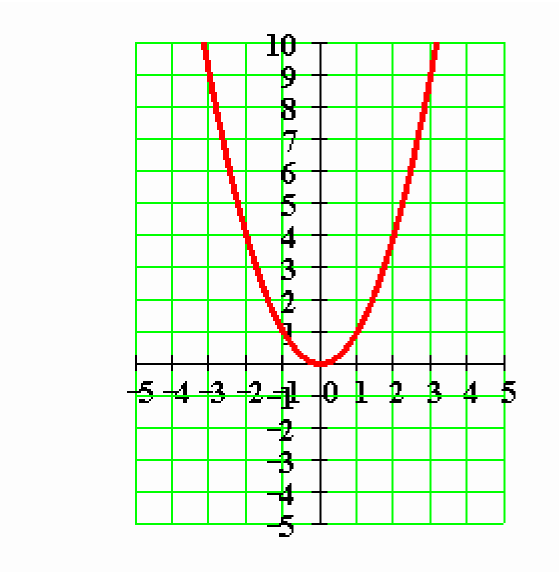
Given $c = 2$ and $f(x) = 3x + 3$, a linear function with domain $(-\infty, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = 3(2) + 3 = 9$ $\lim_{x \to 2^+} f(x) = 9$ $\lim_{x \to 2^-} f(x) = 9$ $\lim_{x \to 2} f(x) = 9$

Problem 2:

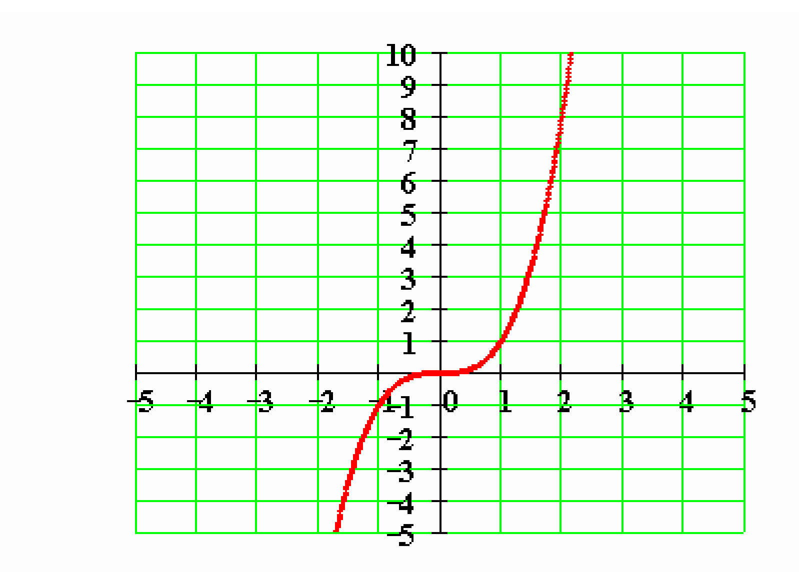
Given $c = 2$ and $f(x) = x^2$, a quadratic function with domain $(-\infty, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = (2)^2 = 4$ $\quad \lim_{x \to 2^-} f(x) = 4$ $\quad \lim_{x \to 2} f(x) = 4$ $\quad \lim_{x \to 2} f(x) = 4$

Problem 3:

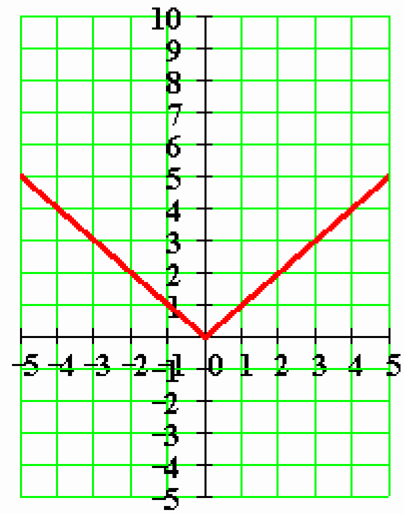
Given $c = 2$ and $f(x) = x^3$, a cubic function with domain $(-\infty, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = (2)^3 = 8$ $\quad \lim_{x \to 2^+} f(x) = 8$ $\quad \lim_{x \to 2^-} f(x) = 8$ $\quad \lim_{x \to 2} f(x) = 8$

Problem 4:

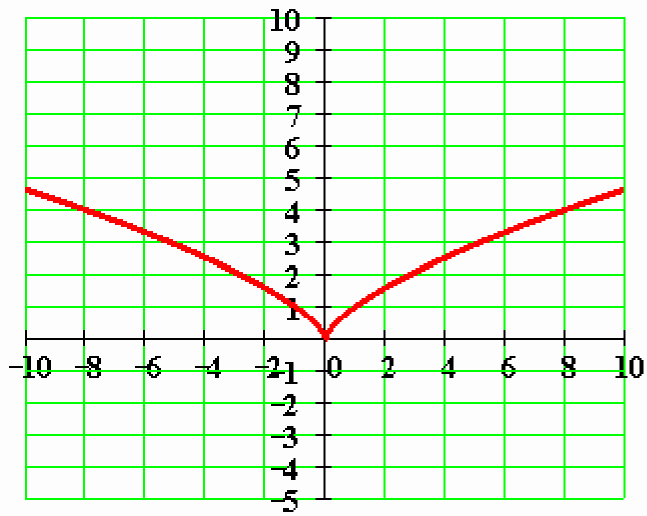
Given $c = -2$ and $f(x) = |x|$, an absolute-value function with domain $(-\infty, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(-2) = |-2| = 2$ $\quad \lim_{x \to -2^+} f(x) = 2$ $\quad \lim_{x \to -2^-} f(x) = 2$ $\quad \lim_{x \to -2} f(x) = 2$

Problem 5:

Given $c = -8$ and $f(x) = x^{2/3}$, a function with domain $(-\infty, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.

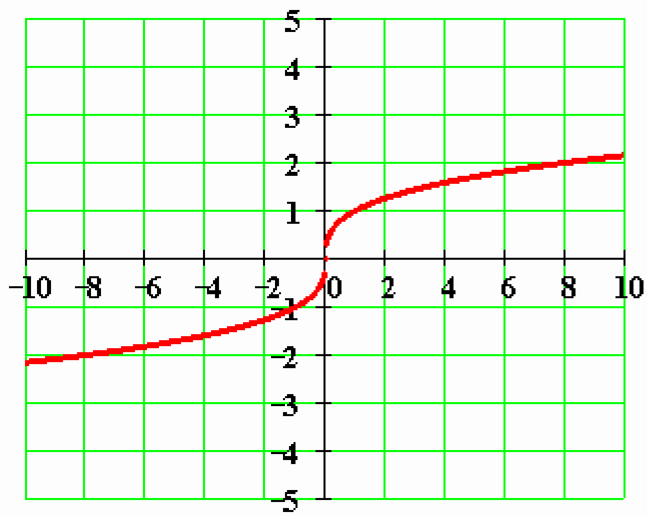


$f(-8) = (-8)^{2/3} = 4$

$$\lim_{x \to -8^+} f(x) = 4 \qquad \lim_{x \to -8^-} f(x) = 4 \qquad \lim_{x \to -8} f(x) = 4$$

Problem 6:

Given $c = -8$ and $f(x) = x^{1/3}$, a function with domain $(-\infty, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.

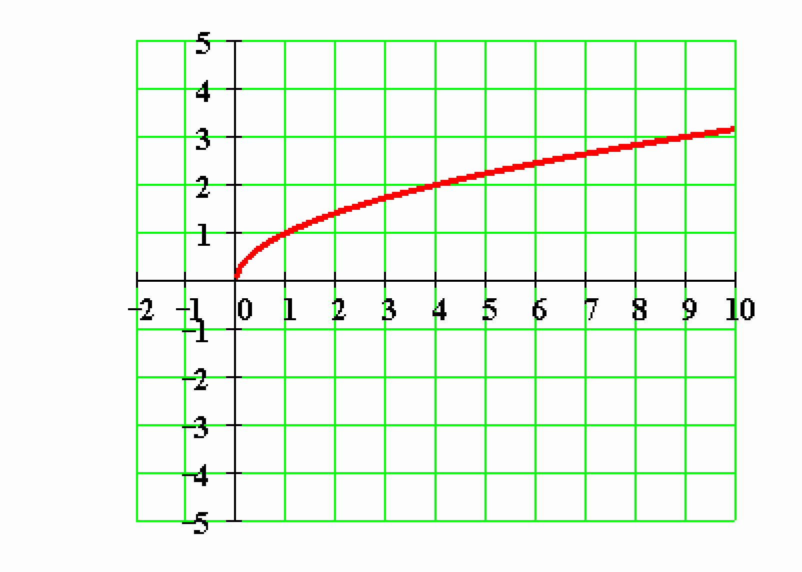


$$f(-8) = (-8)^{1/3} = -2$$

$$\lim_{x \to -8^+} f(x) = -2 \qquad \lim_{x \to -8^-} f(x) = -2 \qquad \lim_{x \to -8} f(x) = -2$$

Problem 7:

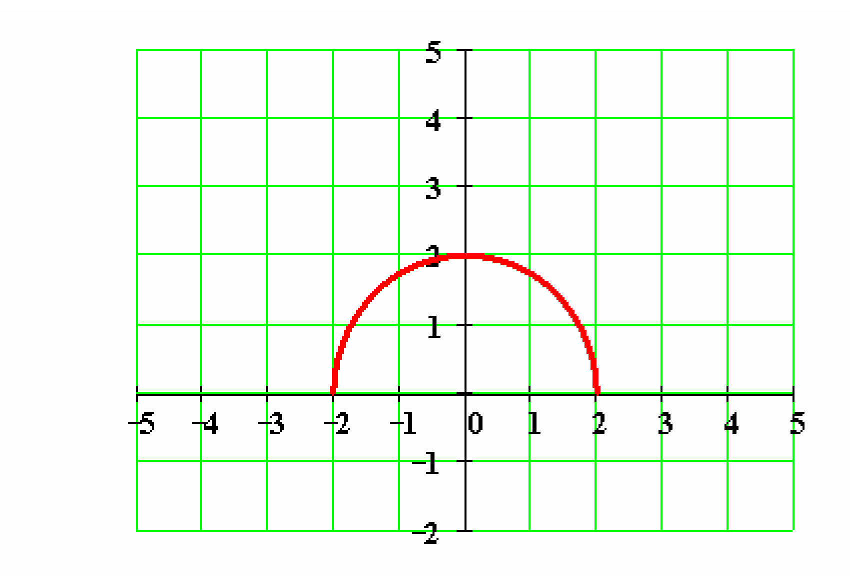
Given $c = 9$ and $f(x) = \sqrt{x}$, a square-root function with domain $[0, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(9) = \sqrt{9} = 3$ $\lim_{x \to 9^+} f(x) = 3$ $\lim_{x \to 9^-} f(x) = 3$ $\lim_{x \to 9} f(x) = 3$

Problem 8:

Given $c = 0$ and $f(x) = \sqrt{4 - x^2}$, a semicircular function with domain $[-2,2]$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.

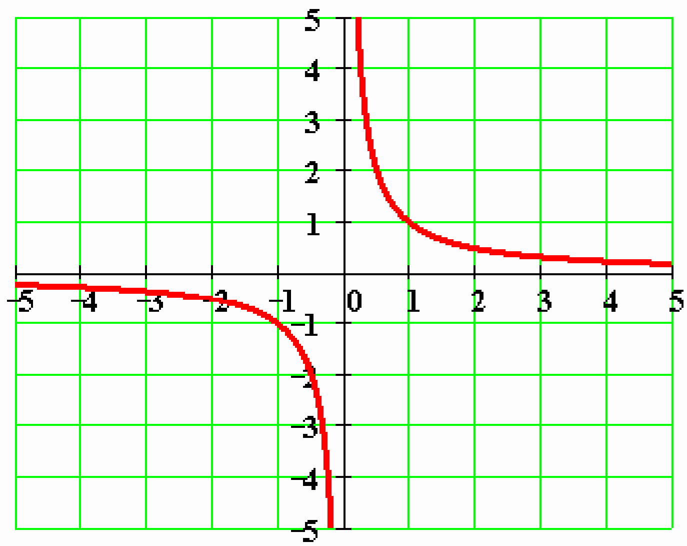


$f(0) = \sqrt{4 - (0)^2} = \sqrt{4} = 2$

$\lim_{x \to 0^+} f(x) = 2$ $\lim_{x \to 0^-} f(x) = 2$ $\lim_{x \to 0} f(x) = 2$

Problem 9:

Given $c = 1$ and $f(x) = \frac{1}{x}$, a rational function (reciprocal function) with domain $\{x \mid x \neq 0\}$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.

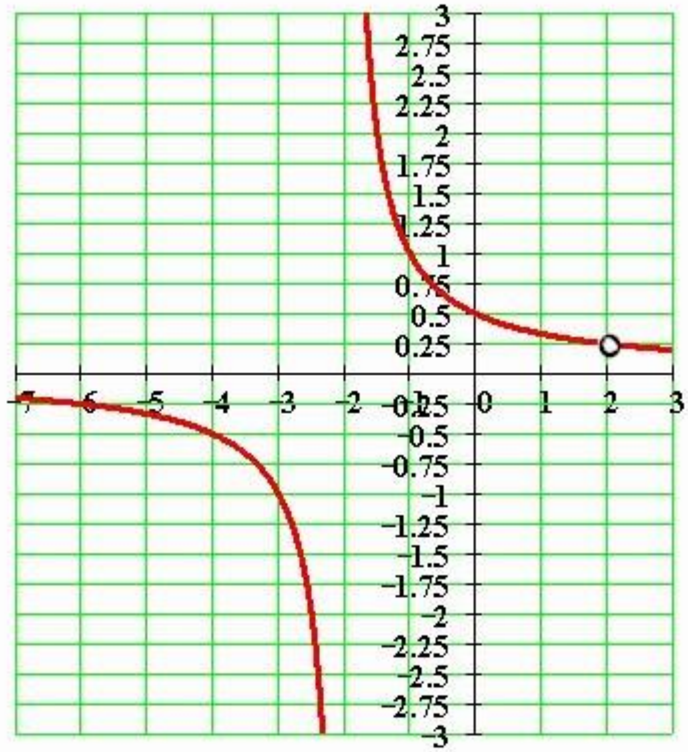


$f(1) = \frac{1}{1} = 1$ $\lim_{x \to 1^+} f(x) = \frac{1}{1} = 1$ $\lim_{x \to 1^-} f(x) = \frac{1}{1} = 1$

$\lim_{x \to 1} f(x) = \frac{1}{1} = 1$

Problem 10:

Given $c = -1$ and $f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)}$, a function with domain $\{x \mid x \neq -2, x \neq 2\}$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



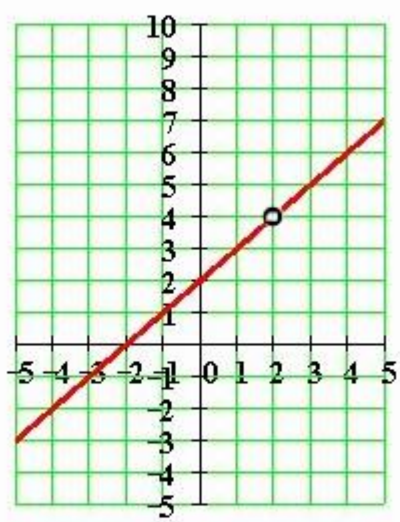
$$f(-1) = \frac{-1-2}{(-1)^2-4} = \frac{-3}{-3} = 1$$

$\lim_{x \to -1^+} f(x) = 1$ $\lim_{x \to -1^-} f(x) = 1$ $\lim_{x \to -1} f(x) = 1$

Problem 11:

Given $c = 1$ and $f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$, a function with domain $\{x \mid x \neq 2\}$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$$f(x) = \frac{1^2 - 4}{1 - 2} = \frac{3}{1} = 3$$

$\lim_{x \to 1^+} f(x) = 3$ $\lim_{x \to 1^-} f(x) = 3$ $\lim_{x \to 1} f(x) = 3$

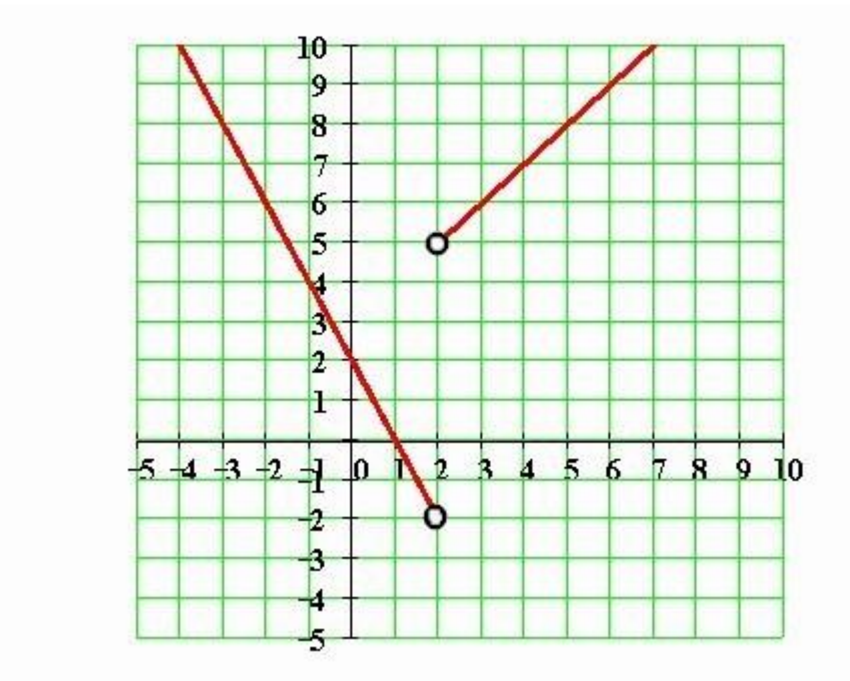
Problem 12:

Given $c = 0$ and $c = 4$ and

$$f(x) = \begin{cases} -2x + 2 & \text{if } x < 2 \\ x + 3 & \text{if } x > 2 \end{cases},$$

a piece-wise defined function with domain $(-\infty, 2) \cup (2, \infty)$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$c = 0$:

$f(0) = -2(0) + 2 = 2$ because 0 is in the domain of Branch 1.

$\lim_{x \to 0^+} f(x) = 2$ $\lim_{x \to 0^-} f(x) = 2$

$\lim_{x \to 0} f(x) = 2$

$c = 4$:

$f(4) = 4 + 3 = 7$ because 4 is in the domain of Branch 2.

$\lim_{x \to 4^+} f(x) = 7$ $\lim_{x \to 4^-} f(x) = 7$

$\lim_{x \to 4} f(x) = 7$

In Problems 13 & 14, note that $f(c)$ is defined, but $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ do not all exist. This usually happens when "c" is in the domain of a function, BUT an endpoint of the domain!

Problem 13:

Given $c = 0$ and $f(x) = \sqrt{x}$, a square-root function with domain $[0, \infty)$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 7**.

$f(0) = \sqrt{0} = 0$

$\lim_{x \to 0^+} f(x) = \sqrt{0} = 0$

$\lim_{x \to 0^-} f(x)$ ***does not exist (DNE)*** because there is NO (real) y-value for the x-values as we approach 0 from the left!

$\lim_{x \to 0} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit!

Problem 14:

Given $c = -2$ and $c = 2$ and $f(x) = \sqrt{4 - x^2}$, a semicircular function with domain $[-2, 2]$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 8**.

$c = -2$:

$f(-2) = \sqrt{4 - (-2)^2} = 0$

$$\lim_{x \to -2^+} f(x) = \sqrt{4-(-2)^2} = 0$$

$\lim_{x \to -2^-} f(x)$ ***does not exist*** because there is NO (real) y-value for the x-values as we approach ***-2*** from the left!

$\lim_{x \to -2} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit!

$c = 2$:

$$f(2) = \sqrt{4-(2)^2} = 0$$

$\lim_{x \to 2^+} f(x)$ ***does not exist*** because there is NO (real) y-value as we approach ***2*** from the right!

$$\lim_{x \to 2^-} f(x) = \sqrt{4-(-2)^2} = 0$$

$\lim_{x \to 2} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit!

In Problems 15 & 16, note that $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ do not exist, and $f(c)$ is undefined! This usually happens when a vertical asymptote exists at "c".

Problem 15:

Given $c = 0$ and $f(x) = \frac{1}{x}$, a rational function (reciprocal function) with domain $\{x \mid x \neq 0\}$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 9**.

$f(0) = \frac{1}{0}$ ***is undefined*** (There is a vertical asymptote at x=0 because the function is reduced to lowest terms!)

$\lim_{x \to 0^+} f(x)$ ***does not exist (DNE)*** because we DO NOT approach a specific y-value as we approach ***0*** from the right.

$\lim_{x \to 0^-} f(x)$ ***does not exist (DNE)*** because we DO NOT approach a specific y-value as we approach ***0*** from the left.

NOTE: Actually, as we approach *0* from the right and left along the x-axis, the y-values are getting infinitely large. However, infinity is NOT considered a SPECIFIC y-value.

$\lim_{x \to 0} f(x)$ ***does not exist (DNE)*** because the right-sided and left-sided limits do not exist.

Problem 16:

Given $c = -2$ and $f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)}$, a function with domain $\{x \mid x \neq -2, x \neq 2\}$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 10**.

$f(-2) = \frac{-2-2}{(-2)^2-4} = \frac{-4}{0}$ ***is undefined*** (There is a vertical asymptote at x = -2)

$\lim_{x \to -2^+} f(x)$ ***does not exist (DNE)*** because we DO NOT approach a specific y-value as we approach ***-2*** from the right.

$\lim_{x \to -2^-} f(x)$ ***does not exist (DNE)*** because we DO NOT approach a specific y-value as we approach ***-2*** from the left.

$\lim_{x \to -2} f(x)$ ***does not exist (DNE)*** because the right-sided and left-sided limits do not exist.

In Problems 17 - 19, note that $f(c)$ might or might not be defined, but $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, and $\lim_{x \to c} f(x)$ all exist and are equal to each other! This usually happens when there is a hole at "c" on the graph of the function.

Problem 17:

Given $c = 2$ and $f(x) = \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)}$, a function with domain $\{x \mid x \neq -2, x \neq 2\}$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 10**.

$f(2) = \frac{2-2}{(2)^2-4} = \frac{0}{0}$ ***is undefined*** (There is a hole at x = 2)

$\lim_{x \to 2^+} f(x) = \frac{1}{4}$ $\lim_{x \to 2^-} f(x) = \frac{1}{4}$ $\lim_{x \to 2} f(x) = \frac{1}{4}$

Problem 18:

Given $c = 2$ and $f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$, a function with domain $\{x \mid x \neq 2\}$. Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 11**.

$f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$ (There is a hole at x=2!)

$\lim_{x \to 2^+} f(x) = 4$ $\lim_{x \to 2^-} f(x) = 4$ $\lim_{x \to 2} f(x) = 4$

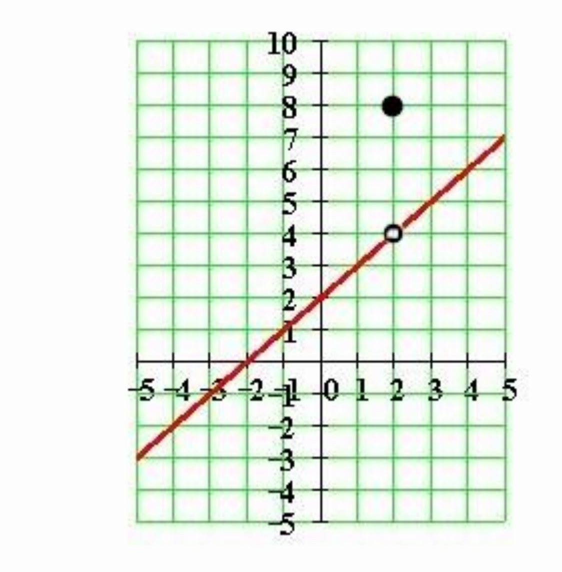
Problem 19:

Given $c = 2$ and

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} & \text{if } x \neq 2 \\ 8 & \text{if } x = 2 \end{cases},$$

a piece-wise defined function with domain $(-\infty, \infty)$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = 8$

$\lim_{x \to 2^+} f(x) = 4$ $\lim_{x \to 2^-} f(x) = 4$ $\lim_{x \to 2} f(x) = 4$

Please note that The Limit and the One-Sided Limits DO NOT equal 8! That is because the second branch of this piece-wise defined function consists only of a point and "approaching" is NOT possible. We can only "approach" using the first branch!

In Problems 20 - 23, note that $\lim_{x \to c} f(x)$ does not exist, $f(c)$ might or might not be defined, but $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ always exist! This usually happens in a piece-wise defined function with a jump discontinuity.

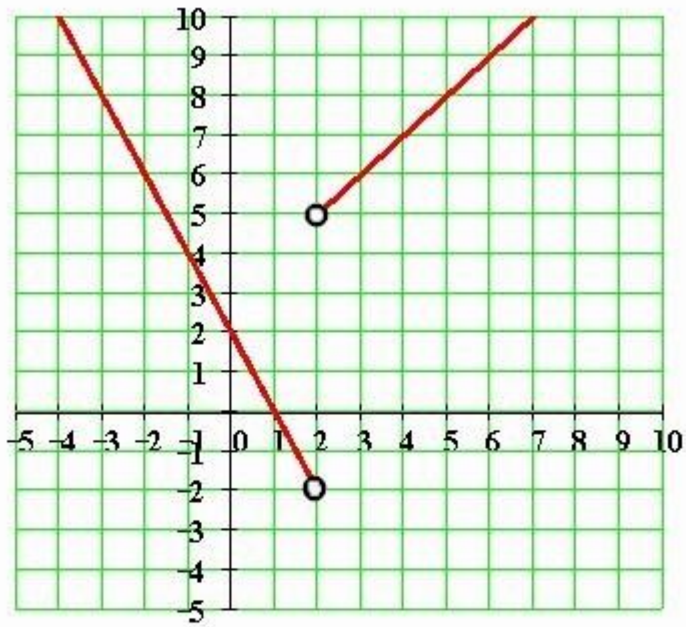
Problem 20:

Given $c = 2$ and

$$f(x) = \begin{cases} -2x + 2 & \text{if } x < 2 \\ x + 3 & \text{if } x > 2 \end{cases},$$

a piece-wise defined function with domain $(-\infty, 2) \cup (2, \infty)$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph in **Problem 12**.



$f(2)$ ***is undefined*** because ***2*** is not in the domain of the function.

$\lim_{x \to 2^+} f(x) = 5$ $\lim_{x \to 2^-} f(x) = -2$

$\lim_{x \to 2} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit.

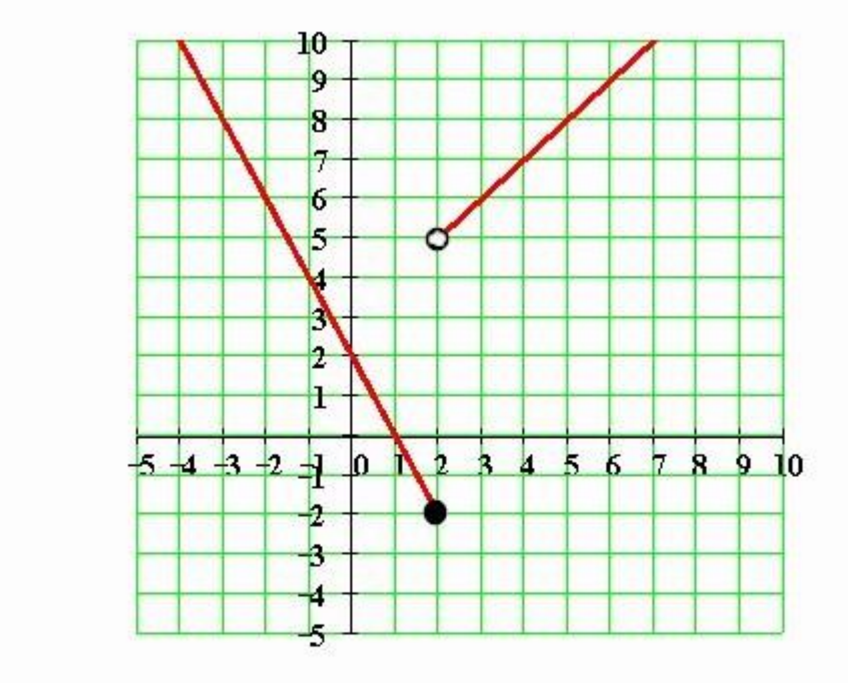
Problem 21:

Given $c = 2$ and

$$f(x) = \begin{cases} -2x + 2 & \text{if } x \le 2 \\ x + 3 & \text{if } x > 2 \end{cases},$$

a piece-wise defined function with domain $(-\infty, \infty)$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = -2(2) + 2 = -2$ because 2 is in the domain of Branch 1.

$\lim_{x \to 2^+} f(x) = 5$ $\lim_{x \to 2^-} f(x) = -2$

$\lim_{x \to 2} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit.

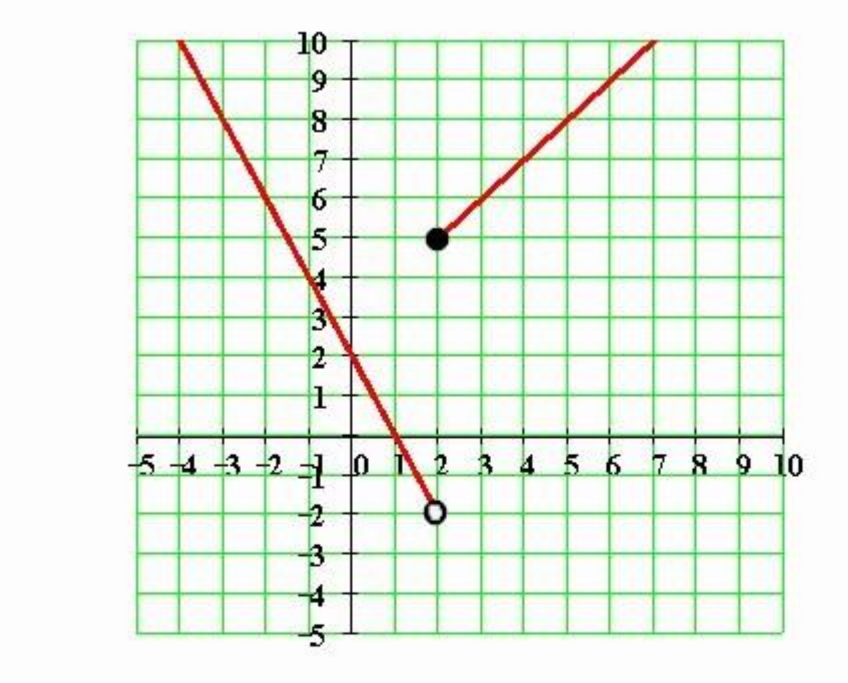
Problem 22:

Given $c = 2$ and

$$f(x) = \begin{cases} -2x + 2 & \text{if } x < 2 \\ x + 3 & \text{if } x \geq 2 \end{cases},$$

a piece-wise defined function with domain $(-\infty, \infty)$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = -2 + 3 = 1$ because ***2*** is in the domain of Branch 2.

$\lim_{x \to 2^+} f(x) = 5$ $\lim_{x \to 2^-} f(x) = -2$

$\lim_{x \to 2} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit.

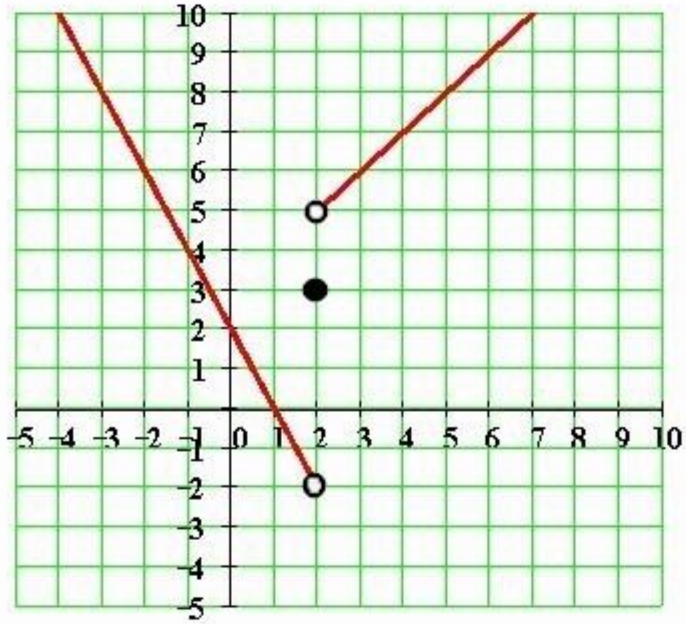
Problem 23:

Given $c = 2$ and

$$f(x) = \begin{cases} -2x + 2 & \text{if } x < 2 \\ x + 3 & \text{if } x > 2 \\ 3 & \text{if } x = 2 \end{cases},$$

a piece-wise defined function with domain $(-\infty, \infty)$.

Calculate $f(c)$ and find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, $\lim_{x \to c} f(x)$ from the graph.



$f(2) = 3$ because $\boldsymbol{2}$ is in the domain of Branch 3.

$\lim_{x \to 2^+} f(x) = 5$ $\lim_{x \to 2^-} f(x) = -2$

$\lim_{x \to 2} f(x)$ ***does not exist (DNE)*** because the right-sided limit DOES NOT equal the left-sided limit.

Please note that The Limit and the One-Sided Limits DO NOT equal *3*! That is because the third branch of this piece-wise defined function consists only of a point and "approaching" is NOT possible. We can only "approach" using the first and second branches!