INTEGRATION BY TABLES AND OTHER INTEGRATION TECHNIQUES

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When a substitution does not simplify an integral so that the previously discussed integration rules can be applied, other methods must be tried. Since integration - unlike differentiation - cannot always be done by a prescribed method, some ingenuity and a lot of practice is required.

Following are two methods that can be tried should you find that an integral cannot be simplified using prescribed techniques.

Integration by Tables

In the "real" word it is often not in anyone's best interest to spend hours finding an antiderivative. Therefore, tables of integration formulas have been designed that allow faster integration.

You can find tables of integration formulas in a "Standard Mathematical Tables" handbook readily available in any bookstore. You can also use an advanced calculator containing a database of integration formulas.

The primary difference between the calculator and the handbook is that the calculator searches through its database to find a fit. Given a handbook, on the other hand, YOU must do the searching using considerable thought and insight and often a substitution of variables.

Unfortunately, the calculator does not always find a fit even though a fit can be found by a person using tables.

In this unit, let's practice using tables with the integrals involving the following inverse trigonometric functions:

a. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$

b. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

c. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Change of Variable

In this case, we will try to substitute all of the variables with ***u***, but the *du*-equation DOES NOT contain the remaining factors of the integrand as was the case for the *Chain Rule*, the *General Power Rule*, and the *Log Rule*.

Problem 1:

Evaluate $\int x\sqrt{2x-1}\,dx$

Since we do not see an exponential or trigonometric expression we know for sure that the *Chain Rule* will not help. What about the *General Power Rule*? Let's use an appropriate *u*-substitution to see!

u = 2x - 1

du/dx = 2

du = 2 dx

We find that the right side of the *du*-equation does not match the remaining factors of the integrand. First of all the constant factor of the integrand is 1. Secondly, the integrand has a factor x, but the *du*-equation does not. Taking care of the constant factor isn't a problem as we have seen in earlier examples. However, there is NOTHING that can be done to place the factor x into the *du*-equation !!! Therefore, the *General Power Rule* cannot be used to find the antiderivative.

Before we resort to integration tables or give up entirely, let's try a different kind of *u*-substitution. That is, we will try to substitute every factor with the variable ***u*** as follows:

Let ***u = 2x - 1***

then ***x = (u + 1)***

du/dx = 2

du = 2 dx

and ***du = dx***

Now, we will rewrite the integral as follows substituting each factor in the original integrand with its *u*-equivalent.

$\int (u + 1)\, u\, (du) = \int u^{1/2}\, (u + 1)\, du$

$= \int (u^{3/2} + u^{1/2})\, du = [\, {}_{2/5}\, u^{5/2} + {}_{2/3}\, u^{3/2}\,] + C$

and we see that we can integrate as follows:

$$\int x\sqrt{2x-1}\,dx = \tfrac{1}{10}(2x-1)^{5/2} + \tfrac{1}{6}(2x-1)^{3/2} + C$$

Therefore, the antiderivative is $$F(x) = \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C$$

Problem 2:

Evaluate $$\int \frac{2x}{(x+1)^2}\,dx$$

Let's rewrite this integral as a product first.

$$\int 2x(x+1)^{-2}\,dx$$

Again, this looks deceivingly like a *General Power Rule*. However, again we see that it is not so!

$u = x + 1$

$du/dx = 1$

$du = dx$

The right side of the *du*-equation does not match the remaining factors of the integrand. First of all the constant factor of the integrand is ***2***. Secondly, the integrand has a factor ***x***, but the *du*-equation does not. Taking care of the constant factor isn't a problem as we have seen in earlier examples. However, there is NOTHING that can be done to place the factor ***x*** into the *du*-equation !!! Therefore, the *General Power Rule* cannot be used to find the antiderivative.

Again, let's try a different kind of *u*-substitution.

Let ***$u = x + 1$***

then ***$x = u - 1$***

$du/dx = 1$

and ***$du = dx$***

Now, we will rewrite the integral as follows substituting each factor in the original integrand with its *u*-equivalent.

$$\int 2\,(u-1)\,u^{-2}\,du = 2\int u^{-2}(u-1)\,du$$

$$= 2\int (u^{-1} - u^{-2})\,du = 2\,(\ln|u| + u^{-1}) + C$$

and we find that the antiderivative can be found as follows:

$$\int \frac{2x}{(x+1)^2}\,dx = 2\ln|x+1| + 2(x+1)^{-1} + C$$

Therefore, the antiderivative is $F(x) = 2\ln|x+1| + 2(x+1)^{-1} + C$

Problem 3:

Evaluate $\int \frac{dx}{\sqrt{4-x^2}}$

Let's rewrite the integral as a product

$$\int (4-x^2)^{-1/2}\,dx$$

If we were to use the *General Power Rule* we would end up with ***du = -2x dx*** containing an ***x*** factor that does not appear in the integrand.

It seems that our new method of using *u*-substitution also does not work! Therefore, let's resort to integration tables.

We quickly find that our integral matches the formula for the inverse sine function.

First of all, let's write the integral as $\int \frac{dx}{\sqrt{(2)^2 - x^2}}$ so that we can clearly see that $a = 2$, $u = x$, and $du = dx$

Now, we can evaluate the integral as follows

$$\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\left(\frac{x}{2}\right) + C$$

Therefore, the antiderivative is $F(x) = \arcsin\left(\frac{x}{2}\right) + C$

Problem 4:

Evaluate $\int \frac{dx}{2+9x^2}$

After rewriting the integral as the product $\int (2+9x^2)^{-2}\,dx$, we again see that the *General Power Rule* or any *u*-substitution for that matter doesn't work.

Again, we'll try to match this integrand to formulas in an integration table. If we rewrite the integral to $\int \frac{dx}{9(\frac{2}{9}+x^2)} = \frac{1}{9}\int \frac{dx}{(\frac{\sqrt{2}}{3})^2 + x^2}$,

with $a = \frac{\sqrt{2}}{3}$, $u = x$, and $du = dx$ we find that we can use the integration formula for the inverse tangent.

Then

$$\frac{1}{9}\int \frac{dx}{(\frac{\sqrt{2}}{3})^2 + x^2} = \frac{1}{9}\left(\frac{1}{\frac{\sqrt{2}}{3}} \arctan\left(\frac{x}{\frac{\sqrt{2}}{3}}\right)\right) + C$$
$$= \frac{1}{3\sqrt{2}} \arctan\left(\frac{3x}{\sqrt{2}}\right) + C$$

Therefore, the antiderivative is $F(x) = \frac{1}{3\sqrt{2}} \arctan\left(\frac{3x}{\sqrt{2}}\right) + C$

Problem 5:

Evaluate $\int \frac{dx}{x\sqrt{4x^2 - 9}}$

Please note that the integration formula for the inverse tangent is $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$.

Observe that the u outside of the radical must match the u inside the radical.

In our example, the u inside the radical actually equals $2x$. Therefore, the u outside the radical must also equal $2x$, but it does not.

Let's approach this integration in two different ways.

(a) Let $u = x$

Change the integral to

$$\int \frac{dx}{x\sqrt{4(x^2 - \frac{9}{4})}} = \frac{1}{\sqrt{4}}\int \frac{dx}{x\sqrt{x^2 - (\frac{3}{2})^2}}$$

then use the inverse secant integration formula with $a = \frac{3}{2}$,

$u = x$, and $du = dx$.

$$\int \frac{dx}{x\sqrt{4(x^2 - \frac{9}{4})}} = \frac{1}{\sqrt{4}} \int \frac{dx}{x\sqrt{x^2 - (\frac{3}{2})^2}} = \frac{1}{2}\left(\frac{1}{\frac{3}{2}} \operatorname{arcsec}\left(\frac{|x|}{\frac{3}{2}}\right)\right) + C$$

$$= \frac{1}{3} \operatorname{arcsec}\left(\frac{2|x|}{3}\right) + C$$

Therefore, the antiderivative is $F(x) = \frac{1}{3} \operatorname{arc\,sec}\left(\frac{2|x|}{3}\right) + C$

(b) Let $u = 2x$

Change the integral to

$$\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{dx}{x\sqrt{(2x)^2 - (3)^2}}$$

$$= \int \frac{2dx}{2x\sqrt{(2x)^2 - (3)^2}}$$

Please note that we multiplied the integrand by a special form of the number ***1***. That is, we multiplied both the numerator and the denominator by ***2*** to get ***u = 2x*** outside the radical !

Now we can use the inverse secant integration formula with $a = 3$, $u = 2x$, and $du = 2\,dx$, where $\frac{1}{2}du = dx$ to get

$$\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2dx}{2x\sqrt{(2x)^2 - (3)^2}}$$

$$= 2\int \frac{dx}{2x\sqrt{(2x)^2 - (3)^2}}$$

$$= \frac{1}{2}(2)\left(\frac{1}{3} \operatorname{arcsec}\left(\frac{2|x|}{3}\right)\right) + C$$

$$= \frac{1}{3} \operatorname{arcsec}\left(\frac{2|x|}{3}\right) + C$$

Therefore, the antiderivative is again $F(x) = \frac{1}{3} \operatorname{arc\,sec}\left(\frac{2|x|}{3}\right) + C$

Problem 6:

Evaluate $\int \frac{dx}{\sqrt{e^{2x}-1}}$

As it stands, this integral does not fit any of the basic integration formulas discussed previously, nor does it seem to fit the three inverse trigonometric formulas.

However using the substitutions $a = 1$, $u = e^x$, and $du = e^x\, dx$,

where $du = u\,dx$ or $\frac{du}{u} = dx$, produces the following

$$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{\sqrt{(e^x)^2-1}} \\ &= \int \frac{du/u}{\sqrt{u^2-1}} \\ &= \int \frac{du}{u\sqrt{u^2-1}} \\ &= \operatorname{arcsec}\frac{|u|}{1} + C\end{aligned}$$

Therefore, we get $\int \frac{dx}{\sqrt{e^{2x}-1}} = \operatorname{arc\,sec}(e^x) + C$

Therefore, the antiderivative is $F(x) = \operatorname{arc\,sec}\left(e^x\right) + C$

Problem 7:

Evaluate $\int \frac{x+2}{\sqrt{4-x^2}}\, dx$

This integral also does not fit any of the basic integration formulas nor any of the three inverse trigonometric formulas.

By splitting the integrand into two parts, however, you can see that the first part can be evaluated with the *General Power Rule* and the second part yields an inverse sine function with $a = 2$, $u = x$, and $du = dx$.

$$\int \frac{x+2}{\sqrt{4-x^2}}\, dx = \int \frac{x}{\sqrt{4-x^2}}\, dx + 2\int \frac{dx}{\sqrt{4-x^2}}$$

Let's evaluate the first integral, that is

$$\int \frac{x}{\sqrt{4-x^2}}\,dx = \int x(4-x^2)^{-1/2}\,dx$$.

Let $u = 4 - x^2$

$du/dx = -2x$

$du = -2x\,dx$

$-\,du = x\,dx$

Therefore, $\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}(2)(4-x^2)^{1/2} + C$.

The entire integral is then evaluated as follows:

$$\int \frac{x+2}{\sqrt{4-x^2}}\,dx = -(4-x^2)^{1/2} + \arcsin\left(\tfrac{x}{2}\right) + C$$

Therefore, the antiderivative is $F(x) = -(4-x^2)^{1/2} + \arcsin\left(\tfrac{x}{2}\right) + C$