$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$ **THE EVALUATION OF DEFINITE INTEGRALS**

Prepared by Ingrid Stewart, Ph.D., College of Southern Nevada
Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

We have finally reached a point, where we can use our knowledge of indefinite integrals and antiderivatives to comfortably evaluate some definite integrals.

Let's review the *Fundamental Theorem of Calculus.*

> If a function ***f*** is continuous on the interval $[a, b]$ and ***F*** is an antiderivative of ***f*** on the interval $[a, b]$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Properties of the Definite Integral

1. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

2. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where ***k*** is any real number

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a \le c \le b$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Please note that some functions simply do not have antiderivatives. If you need to evaluate a definite integral involving a function whose antiderivative cannot be found, the *Fundamental Theorem of Calculus* cannot be applied, and you must resort to an approximation technique.

Two of the more common techniques are the *Trapezoidal Rule* and *Simpson's Rule*, which should be done with a computer program because they can be quite tedious by hand. Most advanced graphing calculators have built-in programs that can be used to approximate the value of a definite integral.

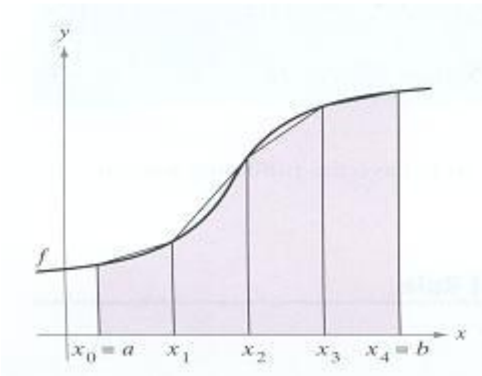
Trapezoidal Rule

Assume that the interval ***[a, b]*** is divided into ***n*** subintervals each of length $\Delta x = \frac{b-a}{n}$.

On each subinterval you then approximate the area of a trapezoid as shown in the picture. The sum of the areas is then considered the value of the definite integral.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_o) + 2f(x_1) + f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)],$$

where $x_o = a$ and $x_1, x_2, \ldots, x_{n-1}, x_n = b$ are the end points of the subintervals.



Observe that the coefficients of the *Trapezoidal Rule* have the following pattern

1 2 2 2 . . . 2 2 2 1

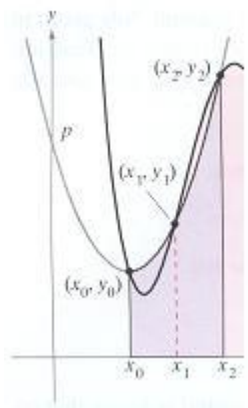
Simpson's Rule

Assume that the interval ***[a, b]*** is divided into ***n*** (must be even) subintervals each of length $\Delta x = \frac{b-a}{n}$.

On each double subinterval you then approximate the area using a polynomial ***p*** of degree less than or equal to two. The sum of the areas is then considered the value of the definite integral.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_o) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + f(x_n)],$$

where $x_o = a$ and $x_1, x_2, \ldots, x_{n-1}, x_n = b$ are the end points of the subintervals.



Observe that the coefficients of *Simpson's Rule* have the following pattern

1 4 2 4 2 . . . 4 2 4 1

Error Analysis

If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be. Following are formulas for estimating the errors ***E*** involved in the use of the *Trapezoidal Rule* and *Simpson's Rule*.

If ***f*** has a continuous second derivative on the interval ***[a, b]***, then the error ***E*** in approximating the value of the definite integral by the *Trapezoidal Rule* is

$$E \le \frac{(b-a)^3}{12n^2}\left(max\left|f''(x)\right|\right), \quad a \le x \le b$$

If ***f*** has a continuous fourth derivative on the interval ***[a, b]***, then the error ***E*** in approximating the value of the definite integral by *Simpson's Rule* is

$$E \le \frac{(b-a)^5}{180n^4}\left(max\left|f^{(4)}(x)\right|\right), \quad a \le x \le b$$

Problem 1:

Evaluate $\int_0^1 (4x^2 - 8x + 1)\,dx$.

Let's find the antiderivatives first!

$$\int (4x^2 - 8x + 1)\,dx = 4\left(\frac{x^{2+1}}{2+1}\right) - 8\left(\frac{x^{1+1}}{1+1}\right) + x + C$$
$$= \tfrac{4}{3}x^3 - 4x^2 + x + C$$

Then

$$\int_0^1 (4x^2 - 8x + 1)dx = \tfrac{4}{3}x^3 - 4x^2 + x\Big]_0^1$$

Please note that the *constant of integration* **C** is not needed in the evaluation of the definite integral since it subtracts out every time during the ***F(b) - F(a)*** calculation! **Try it with the constant of integration and see for yourself!**

$$\int_0^1 (4x^2 - 8x + 1)dx = \overbrace{[\tfrac{4}{3}(1)^3 - 4(1)^2 + 1]}^{F(b)} - \overbrace{[\tfrac{4}{3}(0)^3 - 4(0)^2 + 0]}^{F(a)}$$
$$= \tfrac{4}{3} - 4 + 1$$

and $\int_0^1 (4x^2 - 8x + 1)dx = -\tfrac{5}{3}$

Problem 2:

Evaluate $\int_1^2 \left(x - \frac{1}{x}\right)^2 dx$.

Let's find the antiderivatives first!

$$\int \left(x - \frac{1}{x}\right)^2 dx = \int \left(x^2 - 2 + \frac{1}{x^2}\right)dx$$
$$= \int (x^2 - 2 + x^{-2})dx$$
$$= \left(\frac{x^{2+1}}{2+1}\right) - 2x + \left(\frac{x^{-2+1}}{-2+1}\right) + C$$
$$= \tfrac{1}{3}x^3 - 2x - x^{-1} + C$$
$$= \tfrac{1}{3}x^3 - 2x - \tfrac{1}{x} + C$$

Then

$$\int_1^2 \left(x - \frac{1}{x}\right)^2 dx = \tfrac{1}{3}x^3 - 2x - \tfrac{1}{x}\Big]_1^2$$
$$= [\tfrac{1}{3}(2)^3 - 2(2) - \tfrac{1}{2}] - [\tfrac{1}{3}(1)^3 - 2(1) - \tfrac{1}{1}]$$
$$= \tfrac{8}{3} - 4 - \tfrac{1}{2} - \tfrac{1}{3} + 2 + 1$$

and $\int_1^2 \left(x - \frac{1}{x}\right)^2 dx = \tfrac{5}{6}$

Problem 3:

Evaluate $\int_0^{\pi/6} \frac{1}{4\sec x}\,dx$.

Let's find the antiderivatives first!

$$\int \frac{1}{4\sec x}\,dx = \int \tfrac{1}{4}\cos x\,dx$$
$$= \tfrac{1}{4}\sin x + C$$

Then

$$\int_0^{\pi/6} \frac{1}{4\sec x}\,dx = \tfrac{1}{4}\sin x\,\Big]_0^{\pi/6}$$
$$= \tfrac{1}{4}\left[\sin(\tfrac{\pi}{6}) - \sin(0)\right]$$
$$= \tfrac{1}{4}\left(\tfrac{1}{2} - 0\right)$$

and $\int_0^{\pi/6} \frac{1}{4\sec x}\,dx = \frac{1}{8}$

Problem 4:

Evaluate $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\,dx$.

Let's find the antiderivatives first!

$$\int \frac{1}{\sin^2 x}\,dx = \int \csc^2 x\,dx$$
$$= -\cot x + C$$

Then

$$\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\,dx = -\cot x\,\Big]_{\pi/4}^{\pi/2}$$
$$= \left[-\cot(\tfrac{\pi}{2})\right] - \left[-\cot(\tfrac{\pi}{4})\right]$$
$$= 0 + 1$$

and $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\,dx = 1$

Problem 5:

Evaluate $\int_0^{\pi} 4\cos 4x \, dx$

Let's find the antiderivatives first via the *Chain Rule for Integration*. Using u-substitution, we find

u = 4 x

du/dx = 4

du = 4 dx

Since the right side of the du-equation matches the remaining factors 4 and dx of the integrand exactly, we can now write the integral in terms of ***u*** as follows:

$\int \cos u \, du = \sin u + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int 4\cos 4x \; dx = \sin 4x + C$$

Then

$$\begin{aligned}\int_0^{\pi} 4\cos 4x \, dx &= \sin 4x \Big]_0^{\pi} \\ &= [\sin 4(\pi)] - [\sin 4(0)] \\ &= 0 - 0\end{aligned}$$

and

$$\int_0^{\pi} 4\cos 4x \, dx = 0$$

Problem 6:

Evaluate $\int_{-\pi/4}^{\pi/6} 7 \sec 8x \tan 8x \; dx$

Based on our observations, let's try the *Chain Rule of Integration*! Using *u*-substitution, we find

u = 8 x

du/dx = 8

du = 8 dx

The right side of the *du*-equation does not quite match the remaining factors of the integrand because the constant factor of the integrand is ***7***. This is easily fixed by multiplying both sides of the *du*-equation as follows:

⅞ du = 7 dx

Since the right side of the *du*-equation matches the remaining factors ***7*** and ***dx*** of the integrand exactly, we are now assured that we can use the *Chain Rule* to find the family of antiderivatives.

We proceed to write the integral in terms of ***u*** and evaluate as follows:

$$\tfrac{7}{8} \int \sec u \tan u \, du = \tfrac{7}{8} \sec u + C$$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int 7 \sec 8x \tan 8x \; dx = \tfrac{7}{8} \sec 8x + C$$

Then

$$\begin{aligned} \int_{-\pi/4}^{\pi/6} 7 \sec 8x \tan 8x \; dx &= \tfrac{7}{8} \sec 8x \, \Big]_{-\pi/4}^{\pi/6} \\ &= \tfrac{7}{8} \{ [\sec 8(\tfrac{\pi}{6})] - [\sec 8(-\tfrac{\pi}{4})] \} \\ &= \tfrac{7}{8} (-2 - 1) \end{aligned}$$

and we find that

$$\int_{-\pi/4}^{\pi/6} 7 \sec 8x \tan 8x \; dx = -\tfrac{21}{8}$$

Problem 7:

Evaluate $\int_{-1}^{2} 4x(2x^2+3)^2\ dx$

Let's find the antiderivatives first via the *General Power Rule for Integration*!

$u = 2x^2 + 3$

$du/dx = 4x$

$du = 4x\ dx$

Since the right side of the du-equation matches the remaining factors 4, x, and dx of the integrand exactly, we can now write the integral in terms of ***u*** as follows:

$\int u^2\ du = \frac{1}{3} u^3 + C$

and since the variable of integration is ***x***, we'll convert back to the original variable to find that

$$\int 4x(2x^2+3)^2\ dx = \tfrac{1}{3}(2x^2+3)^3 + C$$

Incidentally, any time we use a *u*-substitution, we can find the value of the definite integral in two ways.

(a) Using the Original Integral:

$$\int_{-1}^{2} 4x(2x^2+3)^2\ dx = \tfrac{1}{3}(2x^2+3)^3\ \Big]_{-1}^{2}$$

$$= \tfrac{1}{3}\{[2(2)^2+3]^3 - [2(-1)^2+3]^3\}$$

$$= \tfrac{1}{3}(1331 - 125)$$

and

$$\int_{-1}^{2} 4x(2x^2+3)^2\ dx = \tfrac{1206}{3} = 402$$

(b) Using the *U*-Substitution:

This is often faster than using the original integral!

Remember that we let ***$u = 2x^2 + 3$*** ! This changes our original limits of integration ***a = -1*** and ***b = 2***!

Given the substitution, the new limits are ***$a = 2(-1)^2 + 3 = 5$*** and ***$b = 2(2)^2 + 3 = 11$***.

Then

$$\tfrac{1}{3}\int_{5}^{11} u^{3}\,du = \tfrac{1}{3}u^{3}\Big]_{5}^{11} = \tfrac{1}{3}(11^{3}-5^{3}) = \tfrac{1206}{3} = 402$$

and

$$\int_{-1}^{2} 4x(2x^{2}+3)^{2}\,dx = \tfrac{1206}{3} = 402$$

Problem 8:

Evaluate $\displaystyle\int_{-\sqrt{8}}^{-\sqrt{5}} \frac{v}{\sqrt{9-v^{2}}}\,dv$

First let's first rewrite the integrand as a product and using exponents

$$\int_{-\sqrt{8}}^{-\sqrt{5}} \frac{v}{\sqrt{9-v^{2}}}\,dv = \int_{-\sqrt{8}}^{-\sqrt{5}} v(9-v^{2})^{-1/2}\,dv$$

Let's first find the antiderivatives via the *General Power Rule for Integration*. Using u-substitution as follows, we find

$$u = 9 - v^{2}$$

$$du/dv = -2v$$

$$du = -2v\,dv$$

The right side of the du-equation contains the remaining factors v and dv of the integrand, but the integrand does not have a constant factor of -2. In this case, we'll divide both sides by the constant factor to get

$$du/_{-2} = v\,dv$$

Since now the right side of the du-equation matches the remaining factors of the integrand exactly, we can write the integral in terms of ***u*** as follows:

$$\int u^{-}\,(du/_{-2}) = \int - u^{-}\,du = -(2u)+C$$

and since the variable of integration is ***v***, we'll convert back to the original variable to find that

$$\int \frac{v}{\sqrt{9-v^{2}}}\,dv = -\sqrt{9-v^{2}} + C$$

Then

$$\int_{-\sqrt{8}}^{-\sqrt{5}} \frac{v}{\sqrt{9-v^2}}\,dv = -\sqrt{9-v^2}\,\Big]_{-\sqrt{8}}^{-\sqrt{5}}$$
$$= [-\sqrt{9-(-\sqrt{5})^2}\,] - [-\sqrt{9-(-\sqrt{8})^2}\,]$$
$$= -2 + 1$$

and

$$\int_{-\sqrt{8}}^{-\sqrt{5}} \frac{v}{\sqrt{9-v^2}}\,dv = -1$$

OR

$$-\int_{1}^{4} u^{1/2}\,du = -u^{1/2}\,\Big]_{1}^{4}$$
$$= -(4^{1/2} - 1^{1/2})$$
$$= -1$$

and

$$\int_{-\sqrt{8}}^{-\sqrt{5}} \frac{v}{\sqrt{9-v^2}}\,dv = -1$$

Problem 9:

(a) Use the definite integral to find the area bounded by $y = 2x$, the x-axis, and the vertical lines $x = -2$ and $x = 3$.

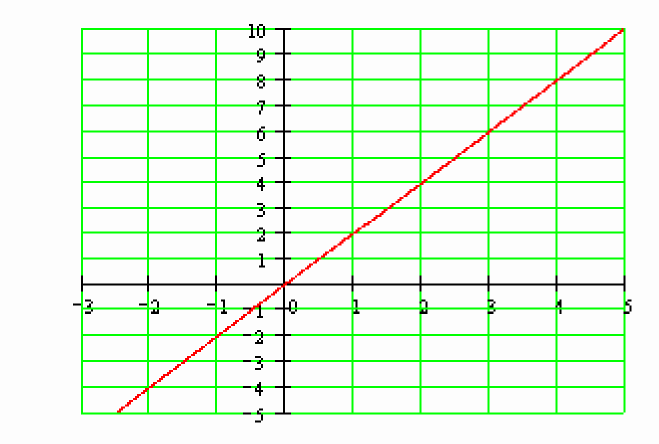
Since we are asked to find an area, we must ensure that all y-values on the interval $[-2,3]$ are nonnegative.

This can usually be checked for any function by first finding the x-intercept of the graph of the function. Then we use test numbers to the right and left of the x-intercepts. If a test number results in a negative y-value we can be sure that all y-values in that subinterval are negative. The same goes for positive y-values.

$$0 = 2x$$
$$x = 0$$

We find the x-intercept to be at the point $(0,0)$. No need to find test numbers since we are dealing with an increasing linear function. Its y-values are negative to the left of the x-intercept and positive to the right of the x-intercept.

See graph below!



In order to find the area on the interval $[-2,3]$, we must do the following

$$A = -\int_{-2}^{0} 2x\,dx + \int_{0}^{3} 2x\,dx$$

Note that we had to multiply the integral by -1 on the interval $[-2,0]$. This counteracts the negativity of the y-values!

Let's work the integrals separately and then add the respective areas.

$$A_1 = -\int_{-2}^{0} 2x\,dx = -x^2 \Big]_{-2}^{0} = -(0)^2 - [-(-2)^2] = 4$$

and $$A_2 = \int_{0}^{3} 2x\,dx = x^2 \Big]_{0}^{3} = (3)^2 - (0)^2 = 9$$

then $$A = A_1 + A_2 = 4 + 9 = 13$$

(b) Evaluate $\int_{-2}^{3} 2x\,dx$

$$\int_{-2}^{3} 2x\,dx = x^2 \Big]_{-2}^{3}$$
$$= (3)^2 - (-2)^2$$

Then

$$\int_{-2}^{3} 2x\,dx = 5$$

Please note that we were NOT asked to find the area bounded by $y = 2x$, the x-axis, and the vertical lines $x = -2$ and $x = 3$. Since there were some negative y-values involved, the value of the integral in (b) is different from the area in (a).

Problem 10:

(a) Use the definite integral to find the area bounded by $f(x) = 4x - x^2$, the x-axis, and the vertical lines $x = 0$ and $x = 4$.

Since we are asked to find an area, we must ensure that all y-values on the interval $[0,4]$ are nonnegative.

Let's first find the x-intercepts of the graph of the function.

$$0 = 4x - x^2$$
$$0 = x(4 - x)$$

and $x = 0$ and $x = 4$

We find the x-intercepts to be at the point $(0,0)$ and at the point $(4,0)$.

Next we use test numbers to the right and the left of the x-intercepts to find out the sign of the y-values on each subinterval.

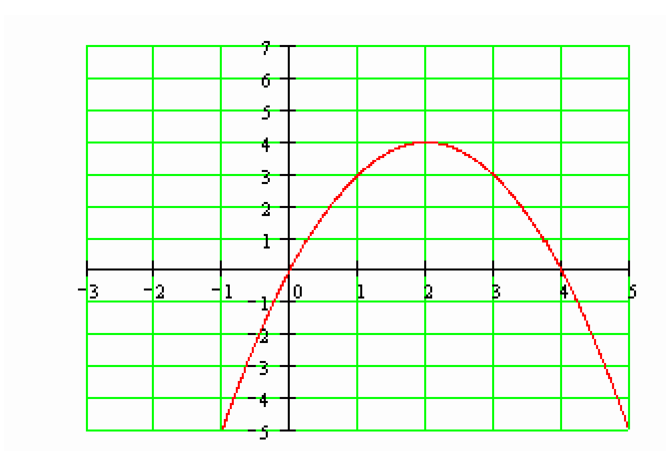
$$f(-1) = 4(-1) - (-1)^2 = -5$$

$$f(2) = 4(2) - (2)^2 = 4$$

$$f(5) = 4(5) - (5)^2 = -5$$

We find that the y-values are nonnegative on the interval $[0,4]$.

See graph below!



Therefore, we can find the area as follows

$$A = \int_0^4 (4x - x^2)\,dx = 2x^2 - \tfrac{1}{3}x^3 \Big]_0^4$$
$$= [2(4)^2 - \tfrac{1}{3}(4)^3] - [2(0)^2 - \tfrac{1}{3}(0)^3]$$
$$= 32 - \tfrac{64}{3}$$

and $A = \frac{32}{3}$

(b) Evaluate $\int_0^4 (4x - x^2)\,dx$

$$\int_0^4 (4x - x^2)\,dx = 2x^2 - \tfrac{1}{3}x^3 \Big]_0^4$$
$$= [2(4)^2 - \tfrac{1}{3}(4)^3] - [2(0)^2 - \tfrac{1}{3}(0)^3]$$
$$= 32 - \tfrac{64}{3}$$

and

$$\int_0^4 (4x - x^2)\,dx = \tfrac{32}{3}$$

Please note that we were NOT asked to find the area bounded by $y = 4x - x^2$, the x-axis, and the vertical lines $x = 0$ and $x = 4$. However, since all y-values were positive on $[0,4]$, the value of the integral in (b) is equal to the area in (a).

Problem 11:

(a) Use the definite integral to find the area bounded by $y = \cos x$, the x-axis, and the vertical lines $x = 0$ and $x = \pi$.

Since we are asked to find an area, we must ensure that all y-values on the interval $[0, \pi]$ are nonnegative.

Let's first find the x-intercepts of the graph of the function.

$$0 = \cos x$$
$$x = \arccos(0)$$
$$x = \frac{\pi}{2}$$

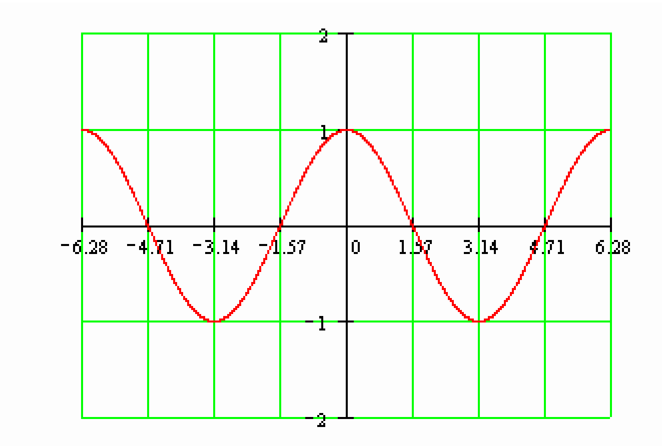
which indicates that the x-coordinates of all x-intercepts of this function are of the form $x = \frac{\pi}{2} + \pi k$, where k is any integer.

On the interval $[0,\pi]$, we find the x-intercept to be at the point $\left(\frac{\pi}{2},0\right)$.

The x-intercept to the left of this point would be at the point $\left(-\frac{\pi}{2},0\right)$ and to the right at the point $\left(\frac{3\pi}{2},0\right)$.

You could use test numbers between the x-intercepts to determine whether the y-values are positive or negative on these subintervals.

For simplicity's sake let's look at the graph of the function.



We find that the y-values are positive on the interval $\left[0,\frac{\pi}{2}\right]$ and negative on the interval $\left[\frac{\pi}{2},\pi\right]$,

In order to find the area on the interval $[0,\pi]$, we must do the following

$$A=\int_0^{\pi/2}\cos x\,dx-\int_{\pi/2}^{\pi}\cos x\,dx$$

Note that we had to multiply the integral on the interval $\left[\frac{\pi}{2},\pi\right]$ by ***-1***. This counteracts the negativity of the y-values!

Let's work the integrals separately and then add the respective areas.

$$A_1=\int_0^{\pi/2}\cos x\,dx=\sin x\,\Big]_0^{\pi/2}=[\sin(\tfrac{\pi}{2})]-[\sin(0)]=1$$

and

$$A_2 = -\int_{\pi/2}^{\pi} \cos x \, dx = -\sin x \Big]_{\pi/2}^{\pi} = [-\sin(\pi)] - [-\sin(\tfrac{\pi}{2})] = 1$$

then $A = A_1 + A_2 = 1 + 1 = 2$

(b) Evaluate $\int_0^{\pi} \cos x \, dx$

$$\int_0^{\pi} \cos x \, dx = \sin x \Big]_0^{\pi}$$
$$= [\sin(\pi)] - [\sin(0)]$$
$$= 0 - 0$$

Then $\int_0^{\pi} \cos x \, dx = 0$

Please note that we were NOT asked to find the area bounded $y = \cos x$, the x-axis, and the vertical lines $x = 0$ and $x = \pi$. Since there were some negative y-values involved, the value of the integral in (b) is different from the area in (a).

Problem 12:

Approximate the value of $\int_1^{1.1} \sin x^2 \, dx$ using the *Trapezoidal Rule* and *Simpson's Rule* with $n = 4$.

Please note that it is NOT possible to find an antiderivative for this integrand!

Let $\Delta x = \dfrac{1.1 - 1}{4} = 0.025$.

Since we are starting the interval at $a = 1$, we will let

$x_0 = 1$, $x_1 = 1.025$, $x_2 = 1.05$, $x_3 = 1.075$, and $x_4 = 1.1$

Trapezoidal Rule

$$\int_1^{1.1} \sin x^2 \, dx \approx \frac{1.1 - 1}{2(4)} \left[\sin(1)^2 + 2\sin(1.025)^2 + 2\sin(1.05)^2 + 2\sin(1.075)^2 + \sin(1.1)^2 \right]$$
$$\approx 0.08910$$

Simpson's Rule

$$\int_1^{1.1} \sin x^2 \, dx \approx \frac{1.1-1}{3(4)}\left[\sin(1)^2 + 4\sin(1.025)^2 + 2\sin(1.05)^2 + 4\sin(1.075)^2 + \sin(1.1)^2\right]$$

$$\approx 0.08911$$