$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$$

CONCAVITY AND INFLECTION POINTS

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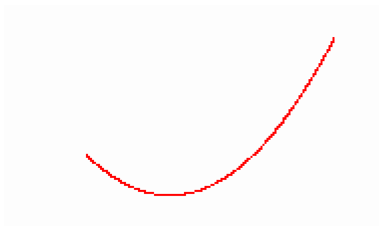
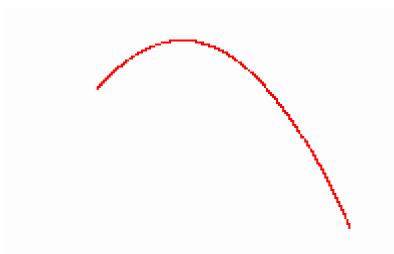
Test for Concavity

The graph of a function is said to be *concave up* on an interval where $f''(x) > 0$.



Likewise, a function is said to be *concave down* on an interval where $f''(x) < 0$.



NOTE: $f''(x) > 0$ means that the slopes of the tangent lines are increasing and $f''(x) < 0$ means that the slopes of the tangent lines are decreasing!

Inflection Points

Points on the graph of a function at which the concavity changes from up to down or vice versa.

If a function has *inflection points*, then they will exist at values of x at which $f''(x) = 0$ or $f''(x)$ does not exist.

> Please note that the converse of the above statement is NOT true. That is, just because we can find values of x at which $f''(x) = 0$ or $f''(x)$ does not exist, does NOT mean that an *inflection point* exists there.

How to find the x-coordinates of POSSIBLE *Inflection Points:*

- Set the **second** derivative equal to 0 and solve.
- If the **second** derivative has a denominator containing a variable, set the denominator equal to 0 and solve.

NOTE:

Solutions must be in the domain of the function to be considered x-coordinates of **possible** *inflection points*.

How to find the x-coordinates of ACTUAL *Inflection Points:*

1. Find all x-coordinate(s) of **possible** *inflection point(s)*.
2. Place the x-coordinate(s) of the **possible** *inflection point(s)* on a number line, thus dividing the number line into two or more intervals.
3. Pick any test value ***t*** from each interval and find ***f ''(t)***.

    a. If ***f ''(t)*** is negative, place minus signs along the number line in the respective interval.
    b. If ***f ''(t)*** is positive, place plus signs along the number line in the respective interval.

4. Use the *Test for Concavity* to determine the concavity on either side of the x-coordinate(s) of the **possible** inflection point(s). If the concavity changes, ONLY then does an x-coordinate of a **possible** inflection point become an x-coordinate of an **actual** inflection point.

Please note that these are the functions from the examples in the previous two sections!

Problem 1:

Given $f(x) = 2x^3 + x^2 - 20x + 4$ with domain $(-\infty, \infty)$, find

- the coordinates of any *inflection points* on the graph of the function

Step 1 - Find all x-coordinate(s) of **possible** *inflection point(s)*

The first derivative is $f'(x) = 6x^2 + 2x - 20$.

The second derivative is $f''(x) = 12x + 2$

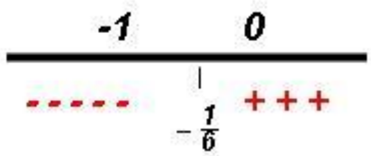
Then

$$12x + 2 = 0$$
$$x = -\tfrac{1}{6}$$

, which is in the domain of the function.

Since the second derivative does not have a denominator containing variables, we find exactly one x-coordinate of a **possible** *inflection point*, namely $-\frac{1}{6}$.

Step 2 and Step 3 - Let's place the x-coordinate of the **possible** *inflection point* on the number line and pick the test numbers 0 and 1 from the respective intervals.



Then $f''(-1) = -10$ and $f''(0) = +2$.

Please note the respective plus and minus signs in the intervals above!

Step 4 - Using the *Test for Concavity* we find that concavity changes from down to up on either side of the x-coordinate of the possible *inflection point*. Therefore, we have found $-\frac{1}{6}$ to be the x-coordinate of an actual *inflection point*.

Its coordinates are $[-\frac{1}{6}, f(-\frac{1}{6})] = (-\frac{1}{6}, \frac{397}{54})$.

- the intervals over which the function is concave up and concave down

  The interval over which the graph of the function is concave up is $(-\frac{1}{6}, \infty)$.

  The interval over which the graph of the function is concave down is $(-\infty, -\frac{1}{6})$.

Problem 2:

Given $f(x) = x^{2/3} - 1$ with domain $(-\infty, \infty)$, find

- the coordinates of any *inflection points* on the graph of the function

  Step 1 - Find all x-coordinate(s) of **possible** *inflection point(s)*

  The first derivative is $f'(x) = \dfrac{2}{3x^{1/3}}$.

  The second derivative is

  $$f''(x) = \frac{2}{3}\left(\frac{-1}{3}x^{-4/3}\right)$$

  and $f''(x) = \dfrac{-2}{9x^{4/3}}$

  Then

  $$0 = \frac{-2}{9x^{4/3}}$$

but $0 \neq -2$ and we find NO **possible** *inflection point.*

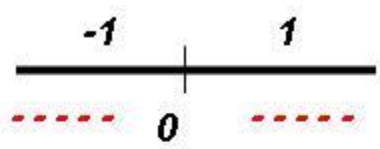
Since the second derivative has a denominator containing a variable, we will set it equal to 0 to find

$9x^{4/3} = 0$

and $x = 0$, which is in the domain of the function.

Therefore, we found the x-coordinate of one **possible** *inflection point*, that is, 0.

Step 2 and Step 3 - Let's place the x-coordinate of the **possible** *inflection point* on the number line and pick the test numbers -1 and 1 from the respective intervals.



Then $f''(-1) = -\frac{2}{9}$ and $f''(1) = -\frac{2}{9}$.

Step 4 - Using the *Test for Concavity* we find that concavity does NOT change on either side of the x-coordinate of the possible *inflection point*. Therefore, we have found that 0 is NOT the x-coordinate of an actual *inflection point*.

- the intervals over which the function is concave up and concave down

The graph is never concave up.

The intervals over which the graph of the function is concave down are $(-\infty, 0)$ and $(0, \infty)$.

Problem 3:

Given $f(x) = \sqrt[3]{x^2 - x - 2}$ with domain $(-\infty, \infty)$, find

- the coordinates of any *inflection points* on the graph of the function

Step 1 - Find all x-coordinate(s) of **possible** *inflection point(s)*

The first derivative is $f'(x) = \dfrac{2x - 1}{3(x^2 - x - 2)^{2/3}}$.

The second derivative is

$$f''(x) = \frac{2\{3(x^2-x-2)^{2/3}\} - (2x-1)\{(2x-1)[3(\frac{2}{3})(x^2-x-2)^{-1/3}]\}}{[3(x^2-x-2)^{2/3}]^2}$$

$$= \frac{6(x^2-x-2)^{2/3} - 2(2x-1)^2(x^2-x-2)^{-1/3}}{9(x^2-x-2)^{4/3}}$$

Since we have to set the second derivative equal to ***0***, it is BEST to write it without negative exponents. Therefore, we will resort to an "old trick" and multiply the right side by a special form of the number ***1***, that is,

$$\frac{(x^2-x-2)^{1/3}}{(x^2-x-2)^{1/3}}$$

$$f''(x) = \frac{6(x^2-x-2)^{2/3} - 2(2x-1)^2(x^2-x-2)^{-1/3}}{9(x^2-x-2)^{4/3}} \cdot \frac{(x^2-x-2)^{1/3}}{(x^2-x-2)^{1/3}}$$

$$= \frac{6(x^2-x-2) - 2(2x-1)^2}{9(x^2-x-2)^{5/3}}$$

$$= \frac{6x^2-2x-12-2(4x^2-4x+1)}{9(x^2-x-2)^{5/3}}$$

and $f''(x) = \frac{-2x^2+2x-14}{9(x^2-x-2)^{5/3}}$

Then

$$0 = \frac{-2x^2+2x-14}{9(x^2-x-2)^{5/3}}$$

$$0 = -2x^2+2x-14$$

$$0 = x^2-x+7$$

Using the *Quadratic Formula*, we get

$x = \frac{1 \pm \sqrt{1-4(1)(7)}}{2(1)} = \frac{1 \pm \sqrt{-27}}{2} = \frac{1 \pm 3i\sqrt{3}}{2}$, which is imaginary and, therefore, can be neither a **possible** nor **actual** *inflection point*.

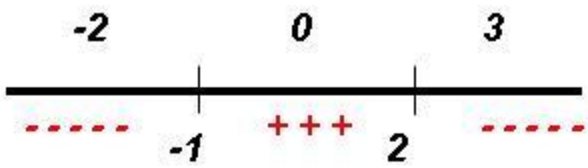
Since the second derivative has a denominator containing a variable, we will set it equal to 0 to find

$$0 = 9(x^2 - x - 2)^{5/3}$$
$$0 = x^2 - x - 2$$
$$(x-2)(x+1) = 0$$

and $x = 2$ and $x = -1$, which are in the domain of the function.

Therefore, we found two x-coordinates of **possible** *inflection points*, that is, -1 and 2.

Step 2 and Step 3 - Let's place the x-coordinates of the **possible** *inflection points* on the number line and pick the test numbers -2, 0, and 3 from the respective intervals.



Then $f''(-2) = \dfrac{-26}{+9(\sqrt[3]{1024})}$

$f''(0) = \dfrac{-14}{-9(\sqrt[3]{1024})}$

and $f''(3) = \dfrac{-26}{+9(\sqrt[3]{1024})}$.

Step 4 - Using the *Test for Concavity* we find that concavity changes on either side of the x-coordinates of the possible *inflection points*. Therefore, we have found -1 and 2 to be the x-coordinates of actual *inflection points*.

The coordinates are $[-1, f(-1)] = (-1,0)$ and $[2, f(2)] = (2,0)$.

- the intervals over which the function is concave up and concave down

The interval over which the graph of the function is concave up is $(-1,2)$.

The intervals over which the graph of the function is concave down are $(-\infty,-1)$ and $(2,\infty)$.

Problem 4:

Given $f(x) = x\sqrt{9-x^2}$ with domain $[-3,3]$, find

- the coordinates of any *inflection points* on the graph of the function

Step 1 - Find all x-coordinate(s) of **possible** *inflection point(s)*

The first derivative is $f'(x) = \dfrac{9-2x^2}{(9-x^2)^{1/2}}$.

The second derivative is

$$f''(x) = \frac{-4x(9-x^2)^{1/2} - (9-2x^2)\{-2x[(\frac{1}{2})(9-x^2)^{-1/2}]\}}{[(9-x^2)^{1/2}]^2}$$
$$= \frac{-4x(9-x^2)^{1/2} + x(9-2x^2)(9-x^2)^{-1/2}}{9-x^2}$$

Since we have to set the second derivative equal to 0, it is BEST to write it without negative exponents. Therefore, we will resort to an "old trick" and multiply the right side by a special form of the number 1, that is,

$$\frac{(9-x^2)^{1/2}}{(9-x^2)^{1/2}}$$

$$f''(x) = \frac{-4x(9-x^2)^{1/2} + x(9-2x^2)(9-x^2)^{-1/2}}{9-x^2} \cdot \frac{(9-x^2)^{1/2}}{(9-x^2)^{1/2}}$$
$$= \frac{-4x(9-x^2) + x(9-2x^2)}{(9-x^2)^{3/2}}$$
$$= \frac{-36x + 4x^3 + 9x - 2x^3}{(9-x^2)^{3/2}}$$

and $f''(x) = \dfrac{2x^3 - 27x}{(9-x^2)^{3/2}}$

Then

$$0=\frac{2x^3-27x}{(9-x^2)^{3/2}}$$
$$0=2x^3-27x$$
$$x(2x^2-27)=0$$

and $x=0$

and $2x^2-27=0$, so that

$x=\sqrt{\frac{27}{2}}\approx 3.67$ and $x=-\sqrt{\frac{27}{2}}\approx -3.67$.

Please note that only *0* is in the domain of the function. Therefore, the other two numbers can be neither possible nor actual inflection points.

Since the second derivative has a denominator containing a variable, we will set it equal to 0 to find

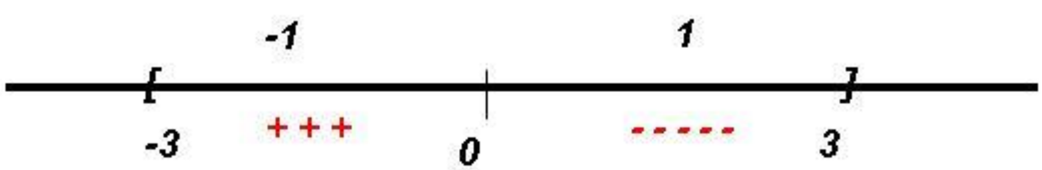
$$0=(9-x^2)^{3/2}$$
$$0=9-x^2$$
$$x^2=9$$

and $x=-3$ and $x=3$, which are in the domain of the function.

Therefore, we found three x-coordinates of **possible** *inflection points*, that is, -3, 0, and 3.

Step 2 and Step 3 - Let's place the x-coordinates of the **possible** *inflection points* on the number line and pick the test numbers -1 and 1 from the respective intervals.

Note that since -3 and 3 are endpoints of the domain we do not need to pick a test number to the left of -3 or to the right of 3! **This also means that these two points could not be actual *inflection points*!**



Then $f''(-1)=\frac{+25}{+\sqrt{512}}$ and $f''(1)=\frac{-25}{+\sqrt{512}}$.

Step 4 - Using the *Test for Concavity* we find that concavity changes on either side of the x-coordinate of the possible *inflection point*. Therefore, we have found 0 to be the x-coordinate of an actual *inflection point*.

The coordinates are $[0, f(0)] = (0,0)$.

- the intervals over which the function is concave up and concave down

The interval over which the graph of the function is concave up is $[-3,0)$.

The interval over which the graph of the function is concave down is $(0,3]$.

Problem 5:

Given $f(x) = x\sqrt{2} - 2\cos x$ with restricted domain $[-2\pi, 2\pi]$, find

- the x-coordinates ONLY of any *inflection points* on the graph of the function

Step 1 - Find all x-coordinate(s) of the **possible** *inflection point(s)*

The first derivative is $f'(x) = \sqrt{2} + 2\sin x$.

The second derivative is $f''(x) = 2\cos x$.

Then

$$0 = 2\cos x$$
$$\cos x = 0$$
$$x = \arccos(0)$$
$$x = 90^\circ \equiv \frac{\pi}{2}$$

The numeric value of cosine is ***0*** only at $90^\circ \equiv \frac{\pi}{2}$ and $270^\circ \equiv \frac{3\pi}{2}$ in the interval $[0, 2\pi)$

Therefore, all solutions are $\frac{\pi}{2} + 2\pi k$ and $\frac{3\pi}{2} + 2\pi k$, where ***k*** is any integer.

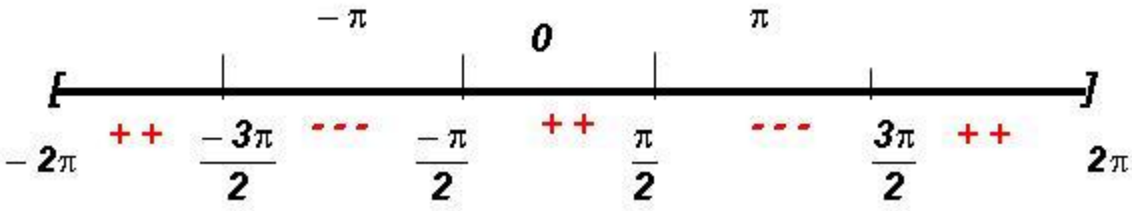
Then the solutions in the interval $[-2\pi, 2\pi]$ are

$\frac{\pi}{2} + 2\pi(0) = \frac{\pi}{2}$ and $\frac{3\pi}{2} + 2\pi(0) = \frac{3\pi}{2}$ and

$\frac{\pi}{2} + 2\pi(-1) = \frac{-3\pi}{2}$ and $\frac{3\pi}{2} + 2\pi(-1) = \frac{-\pi}{2}$

Since the second derivative does not have a denominator containing variables, we find the x-coordinates of four **possible** *inflection points*, namely $\frac{-3\pi}{2}$, $\frac{-\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$ in the domain of the function.

Step 2 and Step 3 - Let's place the x-coordinates of the **possible** *inflection points* on the number line and pick the test numbers -2π, $-\pi$, 0, π and 2π from the respective intervals.



Then $f''(-2\pi) = +2$,

$f''(-\pi) = -2$,

$f''(0) = +2$,

$f''(\pi) = -2$,

and $f''(2\pi) = +2$.

Step 4 - Using the *Test for Concavity* we find that concavity changes on either side of the x-coordinates of the possible *inflection points*. Therefore, we have found $\frac{-3\pi}{2}$, $\frac{-\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$ to be the x-coordinates of actual *inflection points*.

- the intervals over which the function is concave up and concave down

The intervals over which the graph of the function is concave up are

$\left[-2\pi, \frac{-3\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right]$.

The intervals over which the graph of the function is concave down are

$\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Problem 6:

Given $f(x) = (x+2)^3 - 4$ with domain $(-\infty, \infty)$, find

- the coordinates of any *inflection points* on the graph of the function

  Step 1 - Find all x-coordinate(s) of the **possible** *inflection point(s)*

  The first derivative is $f'(x) = 3(x+2)^2$.

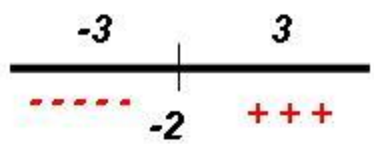
  The second derivative is $f''(x) = 6(x+2)$

  Then $6x + 12 = 0$

  and $x = -2$, which is in the domain of the function.

  Since the second derivative does not have a denominator containing variables, we find exactly one x-coordinate of a **possible** inflection point, namely -2.

  Step 2 and Step 3 - Let's place the x-coordinate of the **possible** *inflection point* on the number line and pick the test numbers -3 and 3 from the respective intervals.



  Then $f''(-3) = -6$ and $f''(3) = 12$.

  Step 4 - Using the *Test for Concavity* we find that concavity changes on either side of the x-coordinate of the possible *inflection point*. Therefore, we have found -2 to be the x-coordinate of an actual *inflection point*.

  The coordinates are $[-2, f(-2)] = (-2, -4)$.

- the intervals over which the function is concave up and concave down

  The interval over which the graph of the function is concave up is $(-2, \infty)$.

  The interval over which the graph of the function is concave down is $(-\infty, 2)$.