$$\lim_{x \to \infty} \int_{2}^{3} \frac{1}{dx}\, dy$$

LIMITS INVOLVING INFINITY

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Theorems

1. If b is any number, then

$\lim_{x \to \infty} b = b$ and $\lim_{x \to -\infty} b = b$

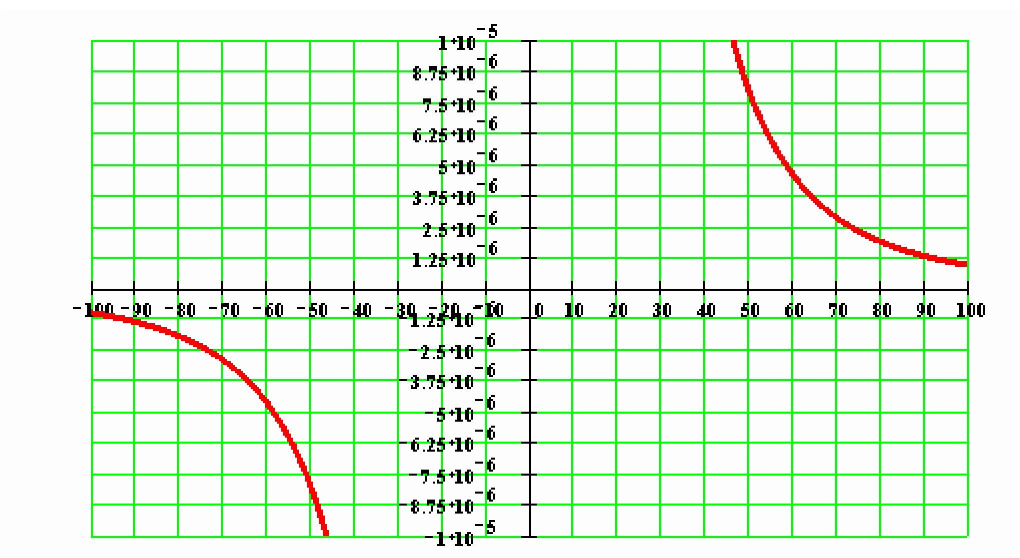
The proof for this is "intuitive." Since we are talking about the function $f(x) = b$, it should be obvious that the y-value will equal b no matter how large $|x|$ becomes.

2. If k is a positive rational number and n is any real number, then

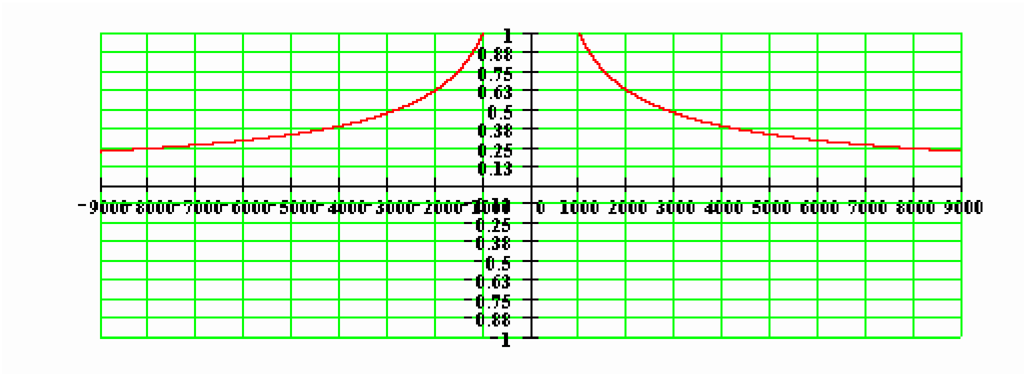
$\lim_{x \to \infty} \frac{n}{x^k} = 0$ and $\lim_{x \to -\infty} \frac{n}{x^k} = 0$

The proof for this again is "intuitive". For example, let's look at the following functions:

$f(x) = \frac{1}{x^3}$, where $n = 1$ and $k = 3$



$f(x) = \frac{1000}{x^{2/3}}$, where $n = 1000$ and $k = \frac{2}{3}$.



Both graphs indicate that as $|x|$ gets larger and larger (***x*** approaches infinity or $x \to \infty$) the y-values get smaller and smaller and ultimately approach ***0***.

Again, be sure that you distinguish between "approaching" and "arriving".

Of course, ***y*** would never equal ***0*** because it isn't even in the range of the functions. However, given a large enough x-value, it will eventually be soooooo small that it might as well be ***0***!

Therefore, we can say $\lim_{x \to \infty} \frac{1}{x^3} = 0$ and $\lim_{x \to -\infty} \frac{1}{x^3} = 0$

as well as $\lim_{x \to \infty} \frac{100}{x^{2/3}} = 0$ and $\lim_{x \to -\infty} \frac{1}{x^{2/3}} = 0$

In this unit, we are only interested in finding $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ for rational functions. To accomplish this, we must do the following:

a. Find the highest power in the denominator.

b. Divide every term in the function by ***x*** raised to this power and reduce.

c. Use the *Limit Laws* from Unit 3 to find *The Limit*.

Finding Horizontal Asymptote in the Graph of Rational Functions

In a Precalculus course we are usually asked to memorize a "recipe" for finding horizontal asymptotes of rational functions. However, this very "recipe" was actually found by observing what happens to the y-value as the x-value approaches positive and negative infinity.

Therefore, $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ actually allows us to find horizontal asymptotes of some functions.

Problem 1:

Find $\lim\limits_{x\to\infty}\dfrac{3x^3-x+1}{6x^3+2x^2-7}$

The highest power in the denominator is ***3***. Therefore, we will divide every term by x^3. Remember that equality is preserved as long as **every** term is divided by x^3.

$$\lim_{x\to\infty}\frac{3x^3-x+1}{6x^3+2x^2-7}=\lim_{x\to\infty}\frac{\dfrac{3x^3}{x^3}-\dfrac{x}{x^3}+\dfrac{1}{x^3}}{\dfrac{6x^3}{x^3}+\dfrac{2x^2}{x^3}-\dfrac{7}{x^3}}$$
$$=\lim_{x\to\infty}\frac{3-\dfrac{1}{x^2}+\dfrac{1}{x^3}}{6+\dfrac{2}{x}-\dfrac{7}{x^3}}$$

Now, the next steps will only be shown in this problem. Thereafter, they will be omitted!

First, we'll use the *Quotient Limit Law*:

$$\lim_{x\to\infty}\frac{3x^3-x+1}{6x^3+2x^2-7}=\frac{\lim\limits_{x\to\infty}\left(3-\dfrac{1}{x^2}+\dfrac{1}{x^3}\right)}{\lim\limits_{x\to\infty}\left(6+\dfrac{2}{x}-\dfrac{7}{x^3}\right)}$$

Next, we'll use the *Sum/Difference Limit Law*:

$$\lim_{x\to\infty}\frac{3x^3-x+1}{6x^3+2x^2-7}=\frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\dfrac{1}{x^2}+\lim\limits_{x\to\infty}\dfrac{1}{x^3}}{\lim\limits_{x\to\infty}6+\lim\limits_{x\to\infty}\dfrac{2}{x}-\lim\limits_{x\to\infty}\dfrac{7}{x^3}}$$

Finally, we use the *Theorems* above to find *The Limit*.

$$\lim_{x\to\infty}\frac{3x^3-x+1}{6x^3+2x^2-7}=\frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\dfrac{1}{x^2}+\lim\limits_{x\to\infty}\dfrac{1}{x^3}}{\lim\limits_{x\to\infty}6+\lim\limits_{x\to\infty}\dfrac{2}{x}-\lim\limits_{x\to\infty}\dfrac{7}{x^3}}$$
$$=\frac{3-0+0}{6+0-0}=\frac{1}{2}$$

Additionally, we have found the equation of the horizontal asymptote of the function $f(x)=\frac{3x^3-x+1}{6x^3+2x^2-7}$. It is $y=\frac{1}{2}$.

Problem 2:

Find $\lim_{x\to\infty}\frac{4x^2}{x^2-4x+3}$

The highest power in the denominator is 2. Therefore, we will divide every term by x^2.

$$\lim_{x\to\infty}\frac{4x^2}{x^2-4x+3}=\lim_{x\to\infty}\frac{\frac{4x^2}{x^2}}{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{3}{x^2}}$$
$$=\lim_{x\to\infty}\frac{4}{1-\frac{4}{x}+\frac{3}{x^2}}$$
$$=\frac{4}{1-0+0}=4$$

Additionally, we have found the equation of the horizontal asymptote of the function

$f(x)=\frac{4x^2}{x^2-4x+3}$. It is $y=4$.

Problem 3:

Find $\lim_{x\to-\infty}\frac{3x^2+1}{2x^4+x^2-1}$

The highest power in the denominator is 4. Therefore, we will divide every term by x^4. Note that we are approaching negative infinity. That is, we are investigating what happens to the y-value given negative x-values.

$$\lim_{x\to-\infty}\frac{3x^2+1}{2x^4+x^2-1}=\lim_{x\to-\infty}\frac{\frac{3x^2}{x^4}+\frac{1}{x^4}}{\frac{2x^4}{x^4}+\frac{x^2}{x^4}-\frac{1}{x^4}}$$
$$=\lim_{x\to-\infty}\frac{\frac{3}{x^2}+\frac{1}{x^4}}{2+\frac{1}{x^2}-\frac{1}{x^4}}$$
$$=\frac{0+0}{2+0-0}=0$$

Note that the same result would occur for $\lim\limits_{x\to\infty} \frac{3x^2+1}{2x^4+x^2-1}$.

As a matter of fact, as long as we are dealing with rational functions, we can say with confidence that $\lim\limits_{x\to-\infty} = \lim\limits_{x\to\infty}$.

Additionally, we have found the equation of the horizontal asymptote of the function

$f(x) = \frac{3x^2+1}{2x^4+x^2-1}$. It is $y=0$.

Problem 4:

Find $\lim\limits_{x\to-\infty} \frac{2x^4+x^2-1}{3x^2+1}$

The highest power in the denominator is ***2***. Therefore, we will divide every term by x^2.

$$\lim_{x\to-\infty} \frac{2x^4+x^2-1}{3x^2+1} = \lim_{x\to-\infty} \frac{\frac{2x^4}{x^2}+\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{3x^2}{x^2}+\frac{1}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{2x^2+1-\frac{1}{x^2}}{3+\frac{1}{x^2}}$$

$$= \frac{\lim\limits_{x\to-\infty} 2x^2 + \lim\limits_{x\to-\infty} 1 - \lim\limits_{x\to-\infty} \frac{1}{x^2}}{\lim\limits_{x\to-\infty} 3 + \lim\limits_{x\to-\infty} \frac{1}{x^2}}$$

Notice that $\lim\limits_{x\to-\infty} 2x^2$ does not exist because as ***x*** approaches negative infinity, the y-value gets larger and larger!

Therefore,

$\lim\limits_{x\to-\infty} \frac{2x^4+x^2-1}{3x^2+1}$ ***does not exist.***.

Note that the same result would occur for $\lim\limits_{x\to\infty} \frac{2x^4+x^2-1}{3x^2+1}$.

Accordingly, the function $f(x) = \frac{2x^4+x^2-1}{3x^2+1}$ does NOT have any horizontal asymptotes.

Problem 5:

Find $\lim_{x \to \infty} \left(5 - \frac{4}{x^3}\right)$

In this case, it is easier to use the *Sum/Difference Limit Law* to write

$$\lim_{x \to \infty} \left(5 - \frac{4}{x^3}\right) = \lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{4}{x^3} = 5 - 0 = 5$$

Problem 6:

Find $\lim_{x \to \pm\infty} \frac{\sin x}{x}$

Since $\sin x$ oscillates between **-*1*** and ***1***, we can write $-1 \le \sin x \le 1$. Now let's divide all terms by x and we find that

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

We know that $\lim_{x \to \pm\infty} \frac{-1}{x} = 0$ and $\lim_{x \to \pm\infty} \frac{1}{x} = 0$, therefore, using the *Squeezing Theorem*, we find that

$\lim_{x \to \pm\infty} \frac{\sin x}{x} = 0$ as can be seen in the graph below.