$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx}\,dy$$

INDEFINITE INTEGRALS AND ANTIDERIVATIVES OF SOME ALGEBRAIC FUNCTIONS

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Through the *Fundamental Theorem of Calculus*, we discovered a close connection between *definite integrals* and *differentiation*.

That is, we can evaluate the *definite integral*

$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ of the function f

by calculating $F(b) - F(a)$, where F is the **antiderivative** of the function f, such that $F'(x) = f(x)$.

In this and the next few units, we will practice finding *families of antiderivatives* F of given functions f , so that we can use the *Fundamental Theorem of Calculus* to evaluate *definite integrals*.

Notation and Vocabulary

1. The symbol $\int f(x)\,dx$ is used to denote the *families of antiderivatives* F of f.

   That is,

   we write $\int f(x)\,dx = F(x) + C$.

2. We call $\int f(x)\,dx$ the **indefinite integral** of f. This symbol asks us to find the antiderivative of f, that is, $F(x) + C$. We pronounce it as *the integral of f(x) with respect to x.*

3. Just like for the *definite integral*, we call the evaluation of $\int f(x)\,dx$ **integration** and $f(x)$ the **integrand**. C is called the **constant of integration**.

For example,

$F(x) = x^2 + C$ is a family of antiderivatives of $f(x) = 2x$ because $F'(x) = 2x = f(x)$.

This is written as an *indefinite integral* as $\int 2x\,dx = x^2 + C$.

NOTE: The *indefinite integral* represents a family of antiderivatives whereas the *definite integral* represents a real number!

For instance, if we write $\int_2^3 2x\,dx$, then we are asked to evaluate a *definite integral* in the interval ***[2, 3]***. That is, we are finding a limit which is a real number!

This limit is calculated using the *Fundamental Theorem of Calculus* with $F(x) = x^2 + C$, $F(b) = 3^2$, and $F(a) = 2^2$.

That is, $\int_2^3 2x\,dx = x^2\Big]_2^3 = 3^2 - 2^2 = 5$.

Please note that the *constant of integration C* is not needed in the evaluation of the definite integral since it subtracts out EVERY time during the *F(b) - F(a)* calculation! Try it with the constant of integration and see for yourself!

Some Basic Integration Formulas (Rules for finding Antiderivatives)

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, for all real numbers ***n*** except ***n ≠ -1***

NOTE: We will integrate x^{-1} in the next unit!

Examples:

$$\int x^5\,dx = \frac{x^{5+1}}{5+1} + C = \tfrac{1}{6}x^6 + C$$

$$\int x^{3/2}\,dx = \frac{x^{3/2+1}}{\frac{3}{2}+1} + C = \frac{x^{5/2}}{\frac{5}{2}} + C = \tfrac{2}{5}x^{5/2} + C$$

$$\int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C$$
$$= -\tfrac{1}{2}x^{-2} + C$$
$$= -\frac{1}{2x^{2}} + C$$

$$\int x\,dx = \frac{x^{1+1}}{1+1} + C$$
$$= \tfrac{1}{2}x^{2} + C$$

2. $\int k\,dx = kx + C$, where ***k*** is any real number

Examples:

a. $\int 2\,dx = 2x + C$

b. $\int -\tfrac{1}{3}\,dx = -\tfrac{1}{3}x + C$

c. $\int dx = x + C$

Properties of the Indefinite Integral

1. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example:

Let $f(x) = x^{5}$ and $g(x) = 2$ then we can write

$$\int (x^{5} + 2)\,dx = \int x^{5}\,dx + \int 2\,dx$$
$$= \frac{x^{5+1}}{5+1} + C_1 + 2x + C_2$$
$$= \tfrac{1}{6}x^{6} + 2x + (C_1 + C_2)$$

NOTE: Assuming that $C_1 + C_2$ equals some value ***C***, it is standard procedure to write

$$\begin{aligned}\int (x^5 + 2)\,dx &= \frac{x^{5+1}}{5+1} + 2x + C \\ &= \tfrac{1}{6}x^6 + 2x + C\end{aligned}$$

2. $\int kf(x)\,dx = k\int f(x)\,dx$, where ***k*** is any real number

Example:

Let $f(x) = x^5$ and $k = -\frac{1}{3}$ then we can write

$$\begin{aligned}\int -\tfrac{1}{3}x^5\,dx &= -\tfrac{1}{3}\int x^5\,dx \\ &= -\tfrac{1}{3}\left(\frac{x^{5+1}}{5+1} + C_1\right) \\ &= -\tfrac{1}{3}\cdot\tfrac{1}{6}x^6 - \tfrac{1}{3}C_1 \\ &= -\tfrac{1}{18}x^6 - \tfrac{1}{3}C_1\end{aligned}$$

NOTE: Assuming that $-\frac{1}{3}C_1$ equals some value ***C***, it is standard procedure to write

$$\begin{aligned}\int -\tfrac{1}{3}x^5\,dx &= -\tfrac{1}{3}\left(\frac{x^{5+1}}{5+1}\right) + C \\ &= -\tfrac{1}{3}\cdot\tfrac{1}{6}x^6 + C \\ &= -\tfrac{1}{18}x^6 + C\end{aligned}$$

Procedures for Fitting Some Integrands to the Basic Integration Formulas

a. change radicals to exponential form

$$\begin{aligned}\int \sqrt[3]{x^2}\,dx &= \int x^{2/3}\,dx \\ &= \frac{x^{2/3+1}}{\frac{2}{3}+1} + C \\ &= \tfrac{3}{5}x^{5/3} + C\end{aligned}$$

b. expand products

$$\int (2x+3)^2\,dx = \int (4x^2+12x+9)\,dx$$
$$= 4\left(\frac{x^{2+1}}{2+1}\right) + 12\left(\frac{x^{1+1}}{1+1}\right) + 9x + C$$
$$= \tfrac{4}{3}x^3 + 6x^2 + 9x + C$$

$$\int x^{1/2}(x-5)\,dx = \int (x^{3/2} - 5x^{1/2})\,dx$$
$$= \left(\frac{x^{3/2+1}}{\frac{3}{2}+1}\right) - 5\left(\frac{x^{1/2+1}}{\frac{1}{2}+1}\right) + C$$
$$= \tfrac{2}{5}x^{5/2} - \tfrac{10}{3}x^{3/2} + C$$

c. rewrite rational expressions as products

$$\int \frac{7}{x^2}\,dx = \int 7x^{-2}\,dx$$
$$= 7\left(\frac{x^{-2+1}}{-2+1}\right) + C$$
$$= -7x^{-1} + C$$
$$= -\frac{7}{x} + C$$

$$\int \frac{x^4 - 2x^2 + 1}{x^2}\,dx = \int (x^4 - 2x^2 + 1)x^{-2}\,dx$$
$$= \int (x^2 - 2 + x^{-2})\,dx$$
$$= \left(\frac{x^{2+1}}{2+1}\right) - 2x + \left(\frac{x^{-2+1}}{-2+1}\right) + C$$
$$= \tfrac{1}{3}x^3 - 2x - x^{-1} + C$$
$$= \tfrac{1}{3}x^3 - 2x - \tfrac{1}{x} + C$$

d. use factoring or long division

$$\int \frac{x^2-9}{x-3}\,dx = \int \frac{(x-3)(x+3)}{x-3}\,dx$$

$$= \int (x+3)\,dx$$

$$= \left(\frac{x^{1+1}}{1+1}\right) + 3x + C$$

$$= \tfrac{1}{2}x^2 + 3x + C$$

$$\int \frac{x^3+4x^2+2x-1}{x+1}\,dx = \int (x^2+3x-1)\,dx$$

$$= \left(\frac{x^{2+1}}{2+1}\right) + 3\left(\frac{x^{1+1}}{1+1}\right) - x + C$$

$$= \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 - x + C$$

Please note

$$\begin{array}{r l}
 & x^2 + 3x - 1 \\
x+1\,\big) & x^3 + 4x^2 + 2x - 1 \\
 & -(x^3 + x^2) \\
\hline
 & 3x^2 + 2x \\
 & -(3x^2 + 3x) \\
\hline
 & -x - 1 \\
 & -(-x-1) \\
\hline
 & 0
\end{array}$$

Problem 1:

Integrate $\int (4x^2 - 8x + 1)\,dx$. Note that "integrate" actually means to find the antiderivative for the function $f(x) = 4x^2 - 8x + 1$!!!

$$\int (4x^2 - 8x + 1)dx = 4\left(\frac{x^{2+1}}{2+1}\right) - 8\left(\frac{x^{1+1}}{1+1}\right) + x + C$$
$$= \tfrac{4}{3}x^3 - 4x^2 + x + C$$

Therefore, the antiderivative is $F(x) = \frac{4}{3}x^3 - 4x^2 + x + C$

To illustrate that differentiation and integration are inverse processes, let's find the derivative of ***F(x)***.

That is, $F'(x) = \tfrac{4}{3}(3x^2) - 4(2x) + 1$

and $F'(x) = 4x^2 - 8x + 1$

We can clearly see that ***F ' (x)*** is equal to the given integrand !!!

Problem 2:

Evaluate $\int \left(\frac{4}{z^6} - \frac{7}{z^4} + z\right) dz$. Note that "evaluate" actually means to find the antiderivative for the function $f(x) = \frac{4}{z^6} - \frac{7}{z^4} + z$!!!

$$\int \left(\frac{4}{z^6} - \frac{7}{z^4} + z\right) dz = \int (4z^{-6} - 7z^{-4} + z)\,dz$$
$$= 4\left(\frac{z^{-6+1}}{-6+1}\right) - 7\left(\frac{z^{-4+1}}{-4+1}\right) + \left(\frac{z^{1+1}}{1+1}\right) + C$$
$$= -\tfrac{4}{5}z^{-5} + \tfrac{7}{3}z^{-3} + \tfrac{1}{2}z^2 + C$$
$$= -\frac{4}{5z^5} + \frac{7}{3z^3} + \frac{1}{2}z^2 + C$$

Therefore, the antiderivative is $F(z) = \frac{4}{5z^5} + \frac{7}{3z^3} + \frac{1}{2}z^2 + C$

Problem 3:

Evaluate $\int(\sqrt{u^3} - \sqrt[5]{u} + 6)du$.

$$\int(\sqrt{u^3} - \sqrt[5]{u} + 6)du = \int(u^{3/2} - u^{1/5} + 6)du$$

$$= \frac{u^{3/2+1}}{\frac{3}{2}+1} - \frac{u^{1/5+1}}{\frac{1}{5}+1} + 6u + C$$

$$= \tfrac{2}{5}u^{5/2} - \tfrac{5}{6}u^{6/5} + 6u + C$$

Therefore, the antiderivative is $F(u) = \frac{2}{5}u^{5/2} - \frac{5}{6}u^{6/5} + 6u + C$

Problem 4:

Evaluate $\int\left(x - \frac{1}{x}\right)^2 dx$.

$$\int\left(x - \frac{1}{x}\right)^2 dx = \int\left(x^2 - 2 + \frac{1}{x^2}\right)dx$$

$$= \int(x^2 - 2 + x^{-2})dx$$

$$= \left(\frac{x^{2+1}}{2+1}\right) - 2x + \left(\frac{x^{-2+1}}{-2+1}\right) + C$$

$$= \tfrac{1}{3}x^3 - 2x - x^{-1} + C$$

$$= \tfrac{1}{3}x^3 - 2x - \tfrac{1}{x} + C$$

Therefore, the antiderivative is $F(x) = \frac{1}{3}x^3 - 2x - \frac{1}{x} + C$

Problem 5:

Evaluate $\int (2x-5)(3x+1)dx$

$$\int (2x-5)(3x+1)dx = \int (6x^2 - 13x - 5)dx$$
$$= 6\left(\frac{x^{2+1}}{2+1}\right) - 13\left(\frac{x^{1+1}}{1+1}\right) - 5x + C$$
$$= 2x^3 - \tfrac{13}{2}x^2 - 5x + C$$

Therefore, the antiderivative is $F(x) = 2x^3 - \dfrac{13}{2}x^2 - 5x + C$

Problem 6:

Evaluate $\int \dfrac{2x^2 - x + 3}{\sqrt{x}}dx$.

$$\int \frac{2x^2 - x + 3}{\sqrt{x}}dx = \int (2x^2 - x + 3)x^{-\frac{1}{2}}dx$$
$$= \int (2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})dx$$
$$= 2\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) - \left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + 3\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$$
$$= \tfrac{4}{5}x^{\frac{5}{2}} - \tfrac{2}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C$$

Therefore, the antiderivative is $F(x) = \dfrac{4}{5}x^{\frac{5}{2}} - \dfrac{2}{3}x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + C$

Problem 7:

Solve the differential equation $f'(x) = 9x^2 + x - 8$ subject to the initial condition $f(0) = 2$.

This means that we must find the antiderivative $f(x)$ of the function $f'(x)$ AND the value of its constant of integration ***C***.

NOTE: Given the initial condition we can actually find the value of ***C*** !!!

$$f(x) = \int (9x^2 + x - 8)\,dx = 9\left(\frac{x^{2+1}}{2+1}\right) + \left(\frac{x^{1+1}}{1+1}\right) - 8x + C$$
$$= 3x^3 + \tfrac{1}{2}x^2 - 8x + C$$

Since $f(x) = 3x^3 + \frac{1}{2}x^2 - 8x + C$, we can now use the initial condition $f(0) = 2$ to determine the value of the constant of integration.

That is, $f(0) = 3(0)^3 + \frac{1}{2}(0)^2 - 8(0) + C = 2$ and we find that $C = 2$.

Therefore, the solution to the differential equation $f'(x) = 9x^2 + x - 8$ is $f(x) = 3x^3 + \frac{1}{2}x^2 - 8x + 2$

Problem 8:

Solve the differential equation $f''(x) = 6x - 4$ subject to the initial conditions $f'(2) = 5$ and $f(2) = 4$.

This also means that we must find the antiderivative $f(x)$ AND the value of its constant of integration ***C***.

Since a second derivative is given, we first have to find $f'(x)$ AND the value of its constant of integration by finding the antiderivative of $f''(x) = 6x - 4$ and using the initial condition $f'(2) = 5$.

$$f'(x) = \int (6x - 4)\,dx = 6\left(\frac{x^{1+1}}{1+1}\right) - 4x + C$$
$$= 3x^2 - 4x + C$$

Since $f'(x) = 3x^2 - 4x + C$, we use the initial condition $f'(2) = 5$ to determine the value of its constant of integration.

That is, $f'(2) = 3(2)^2 - 4(2) + C = 5$.

We find that $C = 1$, therefore, $f'(x) = 3x^2 - 4x + 1$.

Next we have to find $f(x)$ AND the value of its constant of integration by finding the antiderivative of $f'(x) = 3x^2 - 4x + 1$.

$$f(x) = \int (3x^2 - 4x + 1)\,dx = 3\left(\frac{x^{2+1}}{2+1}\right) - 4\left(\frac{x^{1+1}}{1+1}\right) + x + C$$
$$= x^3 - 2x^2 + x + C$$

Since $f(x) = x^3 - 2x^2 + x + C$, we use the initial condition $f(2) = 4$ to determine the value of its constant of integration.

That is, $f(2) = (2)^3 - 2(2)^2 + 2 + C = 4$ and we find that $C = 2$.

Therefore, the solution to the differential equation $f''(x) = 6x - 4$ is $f(x) = x^3 - 2x^2 + x + 2$.