$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx} dy$$

IMPLICIT DIFFERENTIATION

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Review 1:

In **explicitly stated** equations one variable is isolated on one side. For example, $y = 2x^2 - 3$.

In **implicitly stated** equations NO variable is isolated on one side. For example, $y - 2x^2 = -3$.

Review 2:

dy/dx means to differentiate ***y*** with respect to ***x***. For that matter, ***dx/dy*** means that we are supposed to differentiate ***x*** with respect to ***y***.

For example,

(a) if we are asked to find ***dy/dx*** for $y = 2x^2 - 3$, we are actually asked to differentiate ***y*** with respect to ***x***.

That is, ***dy/dx = 4x***.

(b) if the equation $y = 2x^2 - 3$ is implicitly stated as $y - 2x^2 = -3$, we can isolate ***y*** and then find ***dy/dx***.

Unfortunately, it is often too difficult or even impossible to solve for ***y*** in terms of ***x*** when asked to find ***dy/dx***. Consider, for example, $x^2 - y + y^3 = 5$.

What we do in these cases is to consider ***y*** to be a function of ***x*** and then use the *General Power Rule* or the *Chain Rule for Trigonometric Functions* to find ***dy/dx***.

For example, we can then find ***dy/dx*** for $x^2 - y + y^3 = 5$ as follows.

Please note that ***y*** and y^3 will be differentiated using the *General Power Rule* $u' n u^{n-1}$, where ***u*** is a function of ***x***.

That is, their derivative is ***y'*** and $3y' \cdot y^2$, respectively. However, we will be using ***dy/dx*** instead of ***y'*** !!!!!

Therefore, we get $2x - \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$. Please note that when using ***dy/dx***, we'll write it last!

Now, we'll isolate all terms containing a factor of ***dy/dx***

$$-\frac{dy}{dx} + 3y^2 \frac{dy}{dx} = -2x$$

and then we'll factor ***dy/dx*** out of every term

$$\frac{dy}{dx}(-1 + 3y^2) = -2x$$

Finally, we will isolate ***dy/dx*** as follows:

$$\frac{dy}{dx} = \frac{-2x}{-1 + 3y^2} = \frac{2x}{1 - 3y^2}$$

Problem 1:

Find $\frac{dy}{dx}$ given $x^2 + y^2 = 9$.

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

and $\frac{dy}{dx} = -\frac{x}{y}$

You should always write your solutions in lowest terms!

Problem 2:

Find $\frac{dy}{dx}$ given $y^4 + 3y - 4x^3 = 5x + 1$.

$$4y^3 \frac{dy}{dx} + 3\frac{dy}{dx} - 12x^2 = 5$$
$$4y^3 \frac{dy}{dx} + 3\frac{dy}{dx} = 5 + 12x^2$$
$$\frac{dy}{dx}(4y^3 + 3) = 5 + 12x^2$$

and $\frac{dy}{dx} = \frac{5 + 12x^2}{4y^3 + 3}$

Problem 3:

Find $\frac{dy}{dx}$ given $3x^4 - xy = -4y^3$.

We MUST use the *Product Rule* to differentiate xy !

Let's differentiate xy separately. Let $u = x$ and $v = y$, then $u' = 1$ and $v' = \frac{dy}{dx}$. Therefore, the derivative of xy is $y + x\frac{dy}{dx}$.

Finally, differentiating the entire equation, we get

$$12x^3 - \left(y + x\frac{dy}{dx}\right) = -12y^2 \frac{dy}{dx}$$
$$12x^3 - y - x\frac{dy}{dx} = -12y^2 \frac{dy}{dx}$$
$$12y^2 \frac{dy}{dx} - x\frac{dy}{dx} = -12x^3 + y$$
$$\frac{dy}{dx}(12y^2 - x) = -12x^3 + y$$

and $\frac{dy}{dx} = \frac{-12x^3 + y}{12y^2 - x}$

If you wish, you could also write the solution as follows:

$$\frac{dy}{dx} = \frac{y - 12x^3}{12y^2 - x}$$

or multiplying both the numerator and denominator by ***-1***, you may want to write your solution as

$$\frac{dy}{dx} = \frac{12x^3 - y}{x - 12y^2}$$

NOTE: Often in mathematics we shy away from writing terms with negative coefficients first. However, you may write your solution in any way that you see fit!

Problem 4:

Find $\frac{dy}{dx}$ given $y = x^2 \sin y$.

We MUST use the *Product Rule* to differentiate the right side of the equation.

For the *Product Rule* we need $u = x^2$ and $v = \sin y$ so that $u' = 2x$ and $v' = (\cos y)\frac{dy}{dx}$

Finally, differentiating the entire equation, we get

$$\frac{dy}{dx} = 2x\sin y + x^2(\cos y)\frac{dy}{dx}$$
$$\frac{dy}{dx} - x^2(\cos y)\frac{dy}{dx} = 2x\sin y$$
$$\frac{dy}{dx}(1 - x^2\cos y) = 2x\sin y$$

and $\frac{dy}{dx} = \frac{2x\sin y}{1 - x^2\cos y}$

Problem 5:

Find $\frac{dy}{dx}$ given $x = \sin xy$.

Let's differentiate xy separately. Let $u = x$ and $v = y$, then $u' = 1$ and $v' = \frac{dy}{dx}$. Therefore, the derivative of xy is $y + x\frac{dy}{dx}$.

Finally, differentiating the entire equation, we get

$$1 = \left(y + x\frac{dy}{dx}\right)(\cos xy)$$

$$1 = y(\cos xy) + x(\cos xy)\frac{dy}{dx}$$

$$1 - y(\cos xy) = x(\cos xy)\frac{dy}{dx}$$

and $\dfrac{dy}{dx} = \dfrac{1 - y(\cos xy)}{x(\cos xy)}$

Problem 6:

Find $\dfrac{dy}{dx}$ given $y^2 + 1 = x^2 \sec y$.

We MUST use the *Product Rule* to differentiate the equation. That is, we'll let $u = x^2$ and $v = \sec y$ so that $u' = 2x$ and $v' = (\sec y \tan y)\dfrac{dy}{dx}$.

Finally, differentiating the entire equation, we get

$$2y\frac{dy}{dx} = 2x\sec y + x^2(\sec y \tan y)\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - x^2(\sec y \tan y)\frac{dy}{dx} = 2x\sec y$$

$$\frac{dy}{dx}(2y - x^2 \sec y \tan y) = 2x\sec y$$

and $\dfrac{dy}{dx} = \dfrac{2x \sec y}{2y - x^2 \sec y \tan y}$

Problem 7:

Find $\dfrac{dy}{dx}$ given $xy = \tan y$.

We must differentiate xy using the *Product Rule* and the *Chain Rule for Trigonometric Functions* to differentiate $\tan y$.

Let's differentiate xy separately. Let $u = x$ and $v = y$, then $u' = 1$ and $v' = \dfrac{dy}{dx}$. Therefore, the derivative of xy is $y + x\dfrac{dy}{dx}$.

Finally, differentiating the entire equation, we get

$$y + x\frac{dy}{dx} = (\sec^2 y)\frac{dy}{dx}$$
$$x\frac{dy}{dx} - (\sec^2 y)\frac{dy}{dx} = -y$$
$$\frac{dy}{dx}(x - \sec^2 y) = -y$$

and $\frac{dy}{dx} = \frac{-y}{x - \sec^2 y}$

If you wish, you can also write the solution as follows:

$\frac{dy}{dx} = \frac{y}{\sec^2 y - x}$ Note that we multiplied both the numerator and denominator by ***-1***!

Problem 8:

Find $\frac{dy}{dx}$ given $x = \cos y$.

We must differentiate ***cos y*** using the *Chain Rule for Trigonometric Functions*.

Then $1 = (-\sin y)\frac{dy}{dx}$

and we find that $\frac{dy}{dx} = \frac{-1}{\sin y}$ or if you wish $\frac{dy}{dx} = -\frac{1}{\sin y}$

Problem 9:

Find $\frac{dy}{dx}$ given $3x - y^3 = 11x^4$ and evaluate the derivative at the point $(1,-2)$.

$$3 - 3y^2\frac{dy}{dx} = 44x^3$$
$$-3y^2\frac{dy}{dx} = 44x^3 - 3$$
$$\frac{dy}{dx} = \frac{44x^3 - 3}{-3y^2}$$

If you wish, $\frac{dy}{dx} = -\frac{44x^3 - 3}{3y^2}$ or you can multiply the denominator by ***-1*** as follows:

$$\frac{dy}{dx} = \frac{3 - 44x^3}{3y^2}$$

Note that in mathematics we often shy away from writing a term with a negative coefficient first

Finally, at the point $(1,-2)$ we can find $\frac{dy}{dx}$ as follows:

$$\frac{dy}{dx} = \frac{3 - 44(1)^3}{3(-2)^2}$$

and $\frac{dy}{dx} = \frac{-41}{12}$

Problem 10:

Find the *Second Derivative* of $y^2 - 4x^2 = 5$.

First, we must find the *First Derivative* $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} - 8x = 0$$

$$\frac{dy}{dx} = \frac{8x}{2y}$$

and $\frac{dy}{dx} = \frac{4x}{y}$

Now, we'll find the *Second Derivative* $\frac{d^2y}{dx^2}$ by differentiating the *First Derivative*.

We'll use the *Quotient Rule* to find $\frac{d^2y}{dx^2}$ given $u = 4x$, $v = y$, $u' = 4$, and $v' = \frac{dy}{dx}$.

Then $\dfrac{d^2y}{dx^2} = \dfrac{4y - 4x\dfrac{dy}{dx}}{y^2}$

and since $\dfrac{dy}{dx} = \dfrac{4x}{y}$ we can write

$$\frac{d^2y}{dx^2} = \frac{4y - 4x\left(\frac{4x}{y}\right)}{y^2} = \frac{4y - \frac{16x^2}{y}}{y^2}$$

Now, let's write the complex fraction as a simple fraction by multiplying both the numerator and the denominator of the complex fraction by y. Why?

Then $\dfrac{d^2y}{dx^2} = \dfrac{4y - \dfrac{16x^2}{y}}{y^2} \cdot \dfrac{y}{y}$

and $\dfrac{d^2y}{dx^2} = \dfrac{4y^2 - 16x^2}{y^3}$