$$\lim_{x \to \infty} \int_2^3 \frac{1}{dx}\, dy$$

TECHNIQUES FOR FINDING SOME LIMITS ANALYTICALLY

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We already know from the previous section that

$\lim\limits_{x \to c} x = c$ and $\lim\limits_{x \to c} b = b$, where b is any constant

Properties of Limits

Let $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$.

1. Limit of a Sum/Difference :

   $$\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm M$$

2. Limit of $kf(x)$, where k is a Constant:

   $$\lim_{x \to c} [kf(x)] = k \lim_{x \to c} f(x) = kL$$

3. Limit of a Product:

   $$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = L \cdot M$$

4. Limit of a Quotient:

   $$\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} = \frac{L}{M}$$, where $M \neq 0$

5. Limit of $[f(x)]^n$, where $n > 0$ is an integer:

   $$\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n = L^n$$

6. Limit of $\sqrt[n]{f(x)}$, where $n \geq 2$ is an integer and $L \geq 0$ if n is even:

   $$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$$

7. Limit of a Composite Function:

$$\lim_{x \to c} f[g(x)] = f[\lim_{x \to c} g(x)] = f(M)$$

Limits of Polynomial and Rational Functions

1. If p is a polynomial function, then $\lim_{x \to c} p(x) = p(c)$.

2. If $\frac{p}{q}$ is a rational function and $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$.

Limits of Trigonometric Functions

1. Limit of the functions $\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$, and $\cot x$, where $f(c)$ is NOT undefined:

$$\lim_{x \to c} f(x) = f(c)$$

Please note that the domain for $\sin x$ and $\cos x$ is $(-\infty, \infty)$, the domain for $\tan x$ and $\sec x$ is $\{x \mid x \neq \frac{\pi}{2} + k\pi\}$, and the domain for $\csc x$ and $\cot x$ is $\{x \mid x \neq k\pi\}$

2. $$\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 1$$

3. $$\lim_{\theta \to 0} \left(\frac{1 - \cos \theta}{\theta}\right) = 0$$

Check out the documents entitled "The Squeezing Theorem" and "Proofs of Some Trigonometric Limits."

Strategies for Finding Some Limits:

1. Find the domain of the function.

2. Depending on the value of c and the domain, do one of the following:

   a. If we are approaching x-values which are in the domain, but NOT endpoints of the domain of an algebraic or trigonometric function (excluding piecewise-defined functions), then we use one or more of the seven *Limit Laws* or *Theorem 1 and 2* above to find the *Limit*.

b. If we are approaching x-values which are in the domain and are endpoints of the domain of an algebraic or trigonometric function, then the *Limit* does NOT exist because either the right-sided or the left-sided limit does not exist.

c. If we are approaching x-values which are NOT in the domain of a rational function, then the *Limit* may or may not exist.

(1) Reduce any rational expression to lowest terms, if necessary.

(2) Depending on the rational expression, do one of the following:

- If the x-value we are approaching creates ***0*** in the denominator of the ***reduced rational expression***, we are approaching a vertical asymptote in the graph of the function and the *Limit* does NOT exist.

- If the x-value we are approaching DOES NOT create ***0*** in the denominator of the ***reduced rational expression*** OR the expression when reduced does not have a variable in the denominator, we are approaching a hole in the graph of the function and the *Limit* exists.

d. If we are approaching x-values which are NOT in the domain of a trigonometric function, then the *Limit* may or may NOT exist. For example, see the *Special Trigonometric Limits*!!!

Problem 1:

Given $f(x) = 3x + 3$, find $\lim_{x \to 2} f(x)$.

The domain of the function is $(-\infty, \infty)$.

Remember that if ***p*** is a polynomial function, then $\lim_{x \to c} p(x) = p(c)$!

We can find the limit by direct substitution as follows:

$$\lim_{x \to 2} (3x + 3) = 3(2) + 3 = 9$$

Problem 2:

Given $f(x) = x^4 - 3x^2 - 9$, find $\lim_{x \to 2} f(x)$.

The domain of the function is $(-\infty, \infty)$.

$$\lim_{x \to 2} (x^4 - 3x^2 - 9) = (2)^4 - 3(2)^2 - 9 = -5$$

Problem 3:

Given $f(x) = x^2$, find $\lim_{x \to 2} f(x)$

The domain of the function is $(-\infty, \infty)$.

$$\lim_{x \to 2} x^2 = (2)^2 = 4$$

Problem 4:

Given $f(x) = \dfrac{x+3}{x+6}$, find the following limits.

Remember, that if $\frac{p}{q}$ is a rational function and $q(c) \neq 0$, then $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$!

The domain of the function is $\{x \mid x \neq -6\}$.

a. $\lim_{x \to 2} f(x)$

Since ***2*** is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to 2} \left(\frac{x+3}{x+6}\right) = \frac{2+3}{2+6} = \frac{5}{8}$$

b. $\lim_{x \to -6} f(x)$

Since ***-6*** is NOT in the domain of the given function, we first must ensure that the function ***f*** is reduced to lowest terms, which it is.

Since the x-value we are approaching creates ***0*** in the denominator of the ***reduced rational expression***, we are approaching a vertical asymptote in the graph of the function and

$\lim_{x \to -6} f(x)$ ***does not exist*** .

Problem 5:

Given $f(x) = \dfrac{x-2}{x^2-4}$, find the following limits.

The domain of the function is $\{x \mid x \neq -2, x \neq 2\}$**.**

a. $\lim\limits_{x \to 1} f(x)$

Since ***1*** is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to 1} \left(\frac{x-2}{x^2-4}\right) = \frac{1-2}{1^2-4} = \frac{1}{3}$$

b. $\lim\limits_{x \to -2} f(x)$

Since ***-2*** is NOT in the domain of the given function, we first must ensure that the rational expression is reduced to lowest terms.

This is done as follows:

$$\lim_{x \to -2} \left(\frac{x-2}{x^2-4}\right) = \lim_{x \to -2} \left[\frac{x-2}{(x-2)(x+2)}\right] = \lim_{x \to -2} \left(\frac{1}{x+2}\right)$$

Since the x-value we are approaching creates ***0*** in the denominator of the ***reduced rational expression***, we are approaching a vertical asymptote in the graph of the function and

$\lim\limits_{x \to -2} f(x)$ ***does not exist*** .

c. $\lim\limits_{x \to 2} f(x)$

Since ***2*** is NOT in the domain of the given function, we again must ensure that the rational expression is reduced to lowest terms.

$$\lim_{x \to 2} \left(\frac{x-2}{x^2-4}\right) = \lim_{x \to 2} \left[\frac{x-2}{(x-2)(x+2)}\right] = \lim_{x \to 2} \left(\frac{1}{x+2}\right)$$

Since the x-value we are approaching DOES NOT create ***0*** in the denominator of the ***reduced rational expression***, we are approaching a hole in the graph of the function and

$$\lim_{x \to 2} \left(\frac{x-2}{x^2-4} \right) = \lim_{x \to 2} \left(\frac{1}{x+2} \right) = \frac{1}{2+2} = \frac{1}{4}.$$

Problem 6:

Given $f(x) = \dfrac{x^2-4}{x-2}$, find the following limits.

The domain of the function is $\{x \mid x \neq 2\}$.

a. $\lim_{x \to 1} f(x)$

Since ***1*** is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to 1} \left(\frac{x^2-4}{x-2} \right) = \frac{1^2-4}{1-2} = \frac{3}{1} = 3$$

b. $\lim_{x \to 2} f(x)$

Since ***2*** is NOT in the domain of the given function, we again must ensure that the rational expression is reduced to lowest terms.

$$\lim_{x \to 2} \left(\frac{x^2-4}{x-2} \right) = \lim_{x \to 2} \left[\frac{(x-2)(x+2)}{x-2} \right] = \lim_{x \to 2} (x+2)$$

We can see that there is NO denominator containing variables to worry about! Therefore, we are approaching a hole in the graph of the function and

$$\lim_{x \to 2} \left(\frac{x^2-4}{x-2} \right) = \lim_{x \to 2} (x+2) = 2+2 = 4.$$

Problem 7:

Given $f(x) = \sin x$, find $\lim_{x \to \frac{\pi}{2}} f(x)$.

The domain of the function is $(-\infty, \infty)$.

Since $\frac{\pi}{2}$ is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to \frac{\pi}{2}} \sin x = \sin \tfrac{\pi}{2} = 1$$

Problem 8:

Given $f(x) = \cos x$, find $\lim_{x \to \frac{\pi}{3}} f(x)$.

The domain of the function is $(-\infty, \infty)$.

Since $\frac{\pi}{3}$ is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to \frac{\pi}{3}} \cos x = \cos \tfrac{\pi}{3} = \tfrac{1}{2}$$

Problem 9:

Given $f(x) = \tan x$, find the following limits.

The domain of the function is $\{x \mid x \neq \frac{\pi}{2} + k\pi\}$.

a. $\lim_{x \to \frac{\pi}{3}} f(x)$

Since $\frac{\pi}{3}$ is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to \frac{\pi}{3}} \tan x = \tan \tfrac{\pi}{3} = \sqrt{3}$$

b. $\lim_{x \to \frac{\pi}{2}} f(x)$

Since $\frac{\pi}{2}$ is NOT in the domain of the given function, we are approaching a vertical asymptote in the graph of the function and

$\lim_{x \to \frac{\pi}{2}} f(x)$ ***does not exist*** .

Problem 10:

Given $f(x) = \sec x$, find the following limits.

The domain of the function is $\{x \mid x \neq \frac{\pi}{2} + k\pi\}$**.**

a. $\lim_{x \to \frac{\pi}{3}} f(x)$

Since $\frac{\pi}{3}$ is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to \frac{\pi}{3}} \sec x = \sec \frac{\pi}{3} = 2$$

b. $\lim_{x \to \frac{3\pi}{2}} f(x)$

Since $\frac{3\pi}{2}$ is NOT in the domain of the given function, we are approaching a vertical asymptote in the graph of the function and

$\lim_{x \to \frac{3\pi}{2}} f(x)$ ***does not exist*** .

Problem 11:

Given $f(x) = \csc x$, find the following limits.

The domain of the function is $\{x \mid x \neq k\pi\}$.

a. $\lim_{x \to \frac{\pi}{6}} f(x)$

Since $\frac{\pi}{6}$ is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to \frac{\pi}{6}} \csc x = \csc \tfrac{\pi}{6} = 2$$

b. $\lim_{x \to \pi} f(x)$

Since π is NOT in the domain of the given function, we are approaching a vertical asymptote in the graph of the function and

$\lim_{x \to \pi} f(x)$ ***does not exist*** .

Problem 12:

Given $f(x) = \cot x$, find the following limits. .

The domain of the function is $\{x \mid x \neq k\pi\}$.

a. $\lim_{x \to \frac{\pi}{2}} f(x)$

Since $\frac{\pi}{2}$ is in the domain of the given function, we can find the limit by direct substitution as follows:

$$\lim_{x \to \frac{\pi}{2}} \cot x = \cot \tfrac{\pi}{2} = 0$$

b. $\lim_{x \to 0} f(x)$

Since 0 is NOT in the domain of the given function, we are approaching a vertical asymptote in the graph of the function and

$\lim_{x \to 0} f(x)$ ***does not exist*** .

Problem 13:

Given $f(x) = \frac{\sin 5x}{x}$, find $\lim_{x \to 0} f(x)$.

Here we are going to use the special trigonometric limit $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 1$.

However, in our case, we let $\theta = 5x$. Since we only have x in the denominator, we must "supplement" by multiplying by $\frac{5}{5}$ as follows:

$$\lim_{x \to 0} \left(\frac{\sin 5x}{x}\right) = \lim_{x \to 0} \left(\frac{\sin 5x}{x} \cdot \frac{5}{5}\right) = \lim_{x \to 0} \left(5 \cdot \frac{\sin 5x}{5x}\right)$$

Now we are going to use the *2nd Limit Law* to write

$$\lim_{x \to 0} \left(\frac{\sin 5x}{x}\right) = \lim_{x \to 0} \left(5 \cdot \frac{\sin 5x}{5x}\right) = 5 \cdot \lim_{x \to 0} \left(\frac{\sin 5x}{5x}\right)$$

By letting $\theta = 5x$ and observing that $x \to 0$ if and only if $\theta \to 0$, we find that

$$\lim_{x \to 0} \left(\frac{\sin 5x}{x}\right) = 5 \cdot \lim_{x \to 0} \left(\frac{\sin 5x}{5x}\right) = 5 \cdot \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 5(1) = 5$$

Problem 14:

Given $f(x) = \frac{\tan x}{x}$, find $\lim_{x \to 0} f(x)$.

Here we are going to use the *Quotient Identity* to facilitate usage of the special trigonometric limit $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 1$.

$$\lim_{x \to 0} \left(\frac{\tan x}{x}\right) = \lim_{x \to 0} \left(\frac{\sin x}{\cos x} \cdot \frac{1}{x}\right) = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x}\right)$$

Now we are going to use the *3rd Limit Law* to write

$$\lim_{x \to 0} \left(\frac{\tan x}{x}\right) = \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \lim_{x \to 0} \left(\frac{1}{\cos x}\right) = 1\left(\frac{1}{\cos 0}\right) = 1\left(\frac{1}{1}\right) = 1$$

Problem 15:

Given $f(x)=\frac{3(1-\cos x)}{x}$, find $\lim_{x\to 0} f(x)$.

Here we are going to use the special trigonometric limit $\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\right)=0$.

$$\lim_{x\to 0}\frac{3(1-\cos x)}{x}=\lim_{x\to 0}\left(3\cdot\frac{1-\cos x}{x}\right)$$

Now we are going to use the *2nd Limit Law* to write

$$\lim_{x\to 0}\frac{3(1-\cos x)}{x}=\lim_{x\to 0}\left(3\cdot\frac{1-\cos x}{x}\right)$$
$$=3\cdot\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)=3(0)=0$$

Problem 16:

Given $f(x)=\sqrt{x}$, find the following limits.

The domain of the function is $[0,\infty)$.

a. $\lim_{x\to 9} f(x)$

Since we are dealing with neither a polynomial nor a rational function, we must use the *6th Limit Law*.

Please note that ***9*** is in the domain of the given function and NOT an endpoint. Therefore, we can find the limit as follows:

$$\lim_{x\to 9}\sqrt{x}=\sqrt{\lim_{x\to 9} x}=\sqrt{9}=3$$

Notice that the radicand is a polynomial (linear) function, therefore, we were able to find its limit by direct substitution!

b. $\lim_{x\to 0} f(x)$

Since ***0*** is a **left** endpoint of the domain, we can say with confidence that the **left-sided** limit does not exist, which means that

$\lim_{x\to 0} f(x)$ ***does not exist***.

Problem 17:

Given $f(x) = \sqrt{4 - x^2}$, find the following limits.

The domain of the function is $[-2,2]$**.**

a. $\lim_{x \to 0} f(x)$

Since we are dealing with neither a polynomial nor a rational function, we must use *6th Limit Law*.

Please note that ***0*** is in the domain of the given function and NOT an endpoint. Therefore, we can find the limit as follows:

$$\lim_{x \to 0} \sqrt{4 - x^2} = \sqrt{\lim_{x \to 0} (4 - x^2)} = \sqrt{4 - 0} = 2$$

Notice that the radicand is a polynomial (quadratic) function, therefore, we were able to find its limit by direct substitution!

b. $\lim_{x \to -2} f(x)$

Since ***-2*** is a left endpoint of the domain, we can say with confidence that the left-sided limit does not exist, which means that

$\lim_{x \to -2} f(x)$ ***does not exist***.

c. $\lim_{x \to 2} f(x)$

Since ***2*** is a right endpoint of the domain, we can say with confidence that the right-sided limit does not exist, which means that

$\lim_{x \to 2} f(x)$ ***does not exist***.

Problem 18:

Given $f(x) = \sqrt[5]{3x^4 - 5x^2 + 2x}$, find $\lim_{x \to 1} f(x)$.

The domain of the function is $(-\infty, \infty)$**.**

Since we are dealing with neither a polynomial nor a rational function, we must use the *6th Limit Law*.

Please note that ***1*** is in the domain of the given function. Therefore, we can find the limit as follows:

$$\lim_{x\to 1}\sqrt[5]{3x^4-5x^2+2x}=\sqrt[5]{\lim_{x\to 1}(3x^4-5x^2+2x)}$$
$$=\sqrt[5]{3(1)^4-5(1)^2+2(1)}=\sqrt[5]{0}=0$$

Notice that the radicand is a polynomial function, therefore, we were able to find its limit by direct substitution!

Problem 19:

Given $f(x)=\sqrt[5]{\dfrac{x^3-2x^2}{x-3}}$, find $\lim_{x\to 4} f(x)$.

The domain of the function is $\{x\mid x\neq 3\}$.

Since we are dealing with neither a polynomial nor a rational function, we must use the *6th Limit Law*.

Please note that ***4*** is in the domain of the given function and NOT an endpoint. Therefore, we can find the limit as follows:

$$\lim_{x\to 4}\sqrt[5]{\frac{x^3-2x^2}{x-3}}=\sqrt[5]{\lim_{x\to 4}\left(\frac{x^3-2x^2}{x-3}\right)}$$
$$=\sqrt[5]{\frac{4^3-2(4)^2}{4-3}}=\sqrt[5]{32}=2$$

Notice that the radicand is a rational function, therefore, we were able to find its limit by direct substitution!

Problem 20:

Given $f(x)=x^{2/3}$, find $\lim_{x\to -8} f(x)$.

The domain of the function is $(-\infty,\infty)$.

Since we are dealing with neither a polynomial nor a rational function, we must use the *6th Limit Law*.

Please note that ***-8*** is in the domain of the given function and NOT an endpoint. Therefore, we can find the limit as follows:

$$\lim_{x\to -8} x^{2/3}=\lim_{x\to -8}\sqrt[3]{x^2}=\sqrt[3]{\lim_{x\to -8} x^2}=\sqrt[3]{(-8)^2}=\sqrt[3]{64}=4$$

Notice that the radicand is a polynomial (quadratic) function, therefore, we were able to find its limit by direct substitution!