THE DERIVATIVE

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Definition of Secant Line

Given a function ***f***, the *Secant Line* is a line that connects any two points on the graph of ***f***.

Definition of Tangent Line

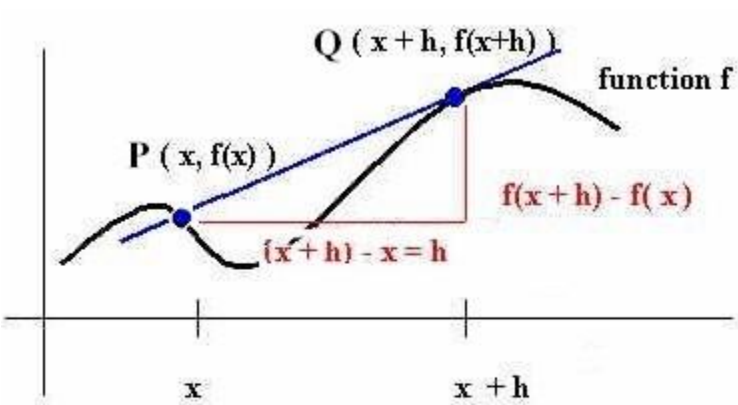
Given a function ***f***, the *Tangent Line* is a line that **TOUCHES** the graph of the function ***f*** at one or more single points.

Note that the emphasis is on the word "touch" !!! A line "crossing" the graph of a function in one or more points is NOT considered a *Tangent Line*.

The Slope of the Secant Line

From Precalculus we know that the SLOPE of the *Secant Line* between two points on the graph of the function ***f*** is the *Difference Quotient*

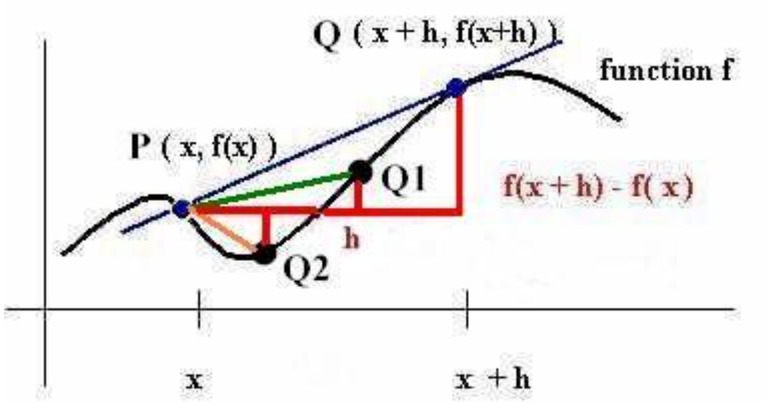
$$m_s = \frac{f(x+h) - f(x)}{h}$$ (see picture below)



The Slope of the Tangent Line

In this course, we are going to investigate what happens to the length of ***h*** as we move the point ***Q*** closer and closer to the point ***P***.

Look at the following picture!



Notice on the graph above, that h gets smaller as the point ***Q*** is moved closer to the point ***P***. As a matter of fact, we define the SLOPE of a *Tangent Line* at ANY point on the graph of the function f as

$$m_t = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note that m_t is often called the *Instantaneous Rate of Change* in application problems!

The Definition of the Derivative

The slope m_t of a line tangent to any point on the graph of the function f is usually denoted as $f'(x)$, which is pronounced *f prime of x.*

That is, we usually write $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, which is called the ***derivative*** of f with respect to (any value) x.

If $f'(x)$ exists, we say that the function f is *differentiable* at the value x or that f has a *derivative* at the value x.

The process of finding $f'(x)$ is called *differentiation*.

$f'(x)$ can also be denoted as:

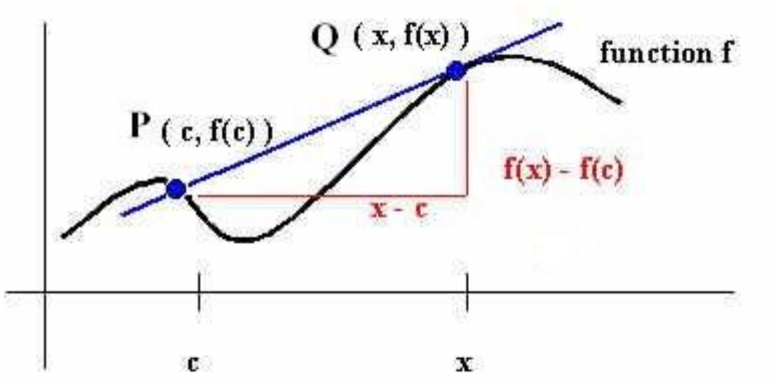
a. $\frac{dy}{dx}$, pronounced *dydx*, which is the derivative of y with respect to x; or

b. $\frac{d}{dx} f(x)$ pronounced as the *derivative of f with respect to x* ; or

c. y', pronounced *y prime*, which is the derivative of ***y*** with respect to ***x***, provided $f(x) = y$.

The Alternative Form of the Derivative at a Point *[c, f(c)]*.

Let's look at another picture illustrating the *Secant Line* and its slope. Please compare this to the picture of the *Difference Quotient* above. Specifically, we changed the coordinates of the points ***P*** and ***Q*** !!!



Using the new notation, we can now rewrite the slope of the *Tangent Line* as follows:

$m_t = f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$, which is the slope of the line tangent to a SPECIFIC point ***[c, f(c)]***.

Problem 1:

Given $f(x) = 3x^2 + 4x - 5$, find $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Let's find the *Difference Quotient* reduced to lowest terms first.

Since

$$\begin{aligned} f(x+h) &= 3(x+h)^2 + 4(x+h) - 5 \\ &= 3(x^2 + 2xh + h^2) + 4x + 4h - 5 \\ &= 3x^2 + 6xh + 3h^2 + 4x + 4h - 5 \end{aligned}$$

we can find the *Difference Quotient* as follows:

$$\frac{f(x+h) - f(x)}{h} = \frac{(3x^2 + 6xh + 3h^2 + 4x + 4h - 5) - (3x^2 + 4x - 5)}{h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{3x^2 + 6xh + 3h^2 + 4x + 4h - 5 - 3x^2 - 4x + 5}{h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{6xh + 3h^2 + 4h}{h}$$

Since we are supposed to reduce the fraction to lowest terms, we will eliminate the ***h*** in the denominator.

$$\frac{f(x+h)-f(x)}{h} = \frac{h(6x + 3h + 4)}{h}$$

$$\frac{f(x+h)-f(x)}{h} = 6x + 3h + 4$$

Finally, we can write

$$\begin{aligned} f'(x) &= \lim_{h\to 0}(6x+3h+4) \\ &= \lim_{h\to 0} 6x + \lim_{h\to 0} 3h + \lim_{h\to 0} 4 \\ &= 6x + 3(0) + 4 \\ &= 6x + 4 \end{aligned}$$

Please note that $\lim_{h\to 0} 6x = 6x$ because we are approaching ***h*** and the term does not contain an ***h***.

Therefore, we treat ***6x*** as a constant using the constant rule $\lim_{x\to 0} b = b$ just like for $\lim_{h\to 0} 4 = 4$.

Further note that $f'(x) = 6x + 4$ happens to be the formula for the slope of a line tangent to any point on the graph of the function $f(x) = 3x^2 + 4x - 5$!!!

Problem 2:

Differentiate the function $f(x) = x - x^2$.

That is, we have to find $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$!!!

Again, let's find the Difference Quotient reduced to lowest terms first.

Since

$$
\begin{aligned}
f(x+h) &= (x+h)-(x+h)^2 \\
&= x+h-(x^2+2xh+h^2) \\
&= x+h-x^2-2xh-h^2
\end{aligned}
$$

we can find the *Difference Quotient* as follows:

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{(x+h-x^2-2xh-h^2)-(x-x^2)}{h} \\
&= \frac{x+h-x^2-2xh-h^2-x+x^2}{h} \\
&= \frac{h-2xh-h^2}{h} = \frac{h(1-2x-h)}{h}
\end{aligned}
$$

and reducing to lowest terms we get

$$\frac{f(x+h)-f(x)}{h} = 1-2x-h$$

Finally, $f'(x) = \lim_{h \to 0} (1-2x-h) = 1-2x$

Again, this happens to be the formula for the slope of a line tangent to any point on the graph of the function $f(x) = x - x^2$!!!

Problem 3:

Find the derivative of $f(x) = \frac{1}{x^2}$.

That is we have to find $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$!!!

The Difference Quotient is $\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

Given a complex fraction, we cannot yet be sure that it is reduced to lowest terms. Since dividing by h means the same as multiplying by the reciprocal of h we can rewrite the *Difference Quotient* as a simple fraction as follows:

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right]$$

Now, before we can say for certain that the *Difference Quotient* is reduced to lowest terms we MUST carry out the subtraction in parentheses. For this we need the common denominator $x^2(x+h)^2$.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}\right]$$

$$= \frac{1}{h}\left[\frac{x^2}{x^2(x+h)^2} - \frac{x^2+2xh+h^2}{x^2(x+h)^2}\right]$$

$$= \frac{1}{h}\left[\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}\right]$$

$$= \frac{1}{h}\left[\frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2}\right]$$

$$= \frac{1}{h}\left[\frac{-2xh-h^2}{x^2(x+h)^2}\right]$$

The last line above tells us that this *Difference Quotient* is not yet reduced to lowest terms since we can factor an ***h*** out of the numerator which allows us to reduce as follows:

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\cdot\frac{h(-2x-h)}{x^2(x+h)^2}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{-2x-h}{x^2(x+h)^2}$$

Finally, $f'(x) = \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$

Again, this happens to be the formula for the slope of a line tangent to any point on the graph of the function $f(x) = \frac{1}{x^2}$!!!

Problem 4:

Find the instantaneous rate of change of the function $f(x) = \sqrt{x}$.

That is, we have to find $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$!!!

The *Difference Quotient* is $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$.

To be sure that the *Difference Quotient* is reduced to lowest terms we must employ a "trick". That is, in the case of radicals we try to multiply the right side of the equation by the conjugate of the numerator as follows:

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$=\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$
$$=\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Finally,

$$f'(x)=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$
$$=\frac{\lim_{h\to 0} 1}{\lim_{h\to 0}\left(\sqrt{x+h}+\sqrt{x}\right)}$$
$$=\frac{\lim_{h\to 0} 1}{\sqrt{\lim_{h\to 0}(x+h)}+\sqrt{\lim_{h\to 0} x}}$$
$$=\frac{1}{\sqrt{x+0}+\sqrt{x}}$$

and $f'(x)=\frac{1}{2\sqrt{x}}$

Again, this happens to be the formula for the slope of a line tangent to any point on the graph of the function $f(x)=\sqrt{x}$!!!

Problem 5:

a. Find the slope of the line tangent to the point $(-1,2)$ on the function $f(x)=x^3-3x$ using the *Difference Quotient* $\frac{f(x+h)-f(x)}{h}$.

Remember that $f'(x)$ is the slope of a line tangent to any point on the graph of the function $f(x)=x^3-3x$.

Since

$$f(x+h)=(x+h)^3-3(x+h)$$
$$=x^3+3x^2h+3xh^2+h^3-3x-3h,$$

we can find the *Difference Quotient* as follows:

$$\frac{f(x+h)-f(x)}{h}=\frac{(x^3+3x^2h+3xh^2+h^3-3x-3h)-(x^3-3x)}{h}$$
$$=\frac{x^3+3x^2h+3xh^2+h^3-3x-3h-x^3+3x}{h}$$
$$=\frac{3x^2h+3xh^2+h^3-3h}{h}$$
$$=3x^2+3xh+h^2-3$$
.

Finally, $f'(x)=\lim_{h\to 0}(3x^2+3xh+h^2-3)=3x^2-3$. **This is the formula for the slope of the line tangent to any point on the graph of the function $f(x)=x^3-3x$.**

However, we are supposed to find $f'(-1)$, which is the slope of the line tangent to the point $(-1,2)$ on the function $f(x)=x^3-3x$.

We find that $f'(-1)=3(-1)^2-3=0$.

It is $m_t=0$ at the point $(-1,2)$.

b. Find the slope of the line tangent to the point $(-1,2)$ on the function $f(x)=x^3-3x$ using the alternative form of the derivative, that is, $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$.

Please compare this to (a) above, where we used the *Difference Quotient*. Actually, when finding the slope of the *Tangent Line* at a particular point, it is much easier to use the alternative form of the derivative.

Let's find $f'(-1)=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}$, that is, the slope of the line tangent to the point $(-1,2)$ on the function $f(x)=x^3-3x$.

We know that $f(-1)=(-1)^3-3(-1)=2$.

Therefore, we can write

$$f'(-1)=\lim_{x\to -1}\frac{(x^3-3x)-2}{x+1}=\lim_{x\to -1}\frac{x^3-3x-2}{x+1}$$

Using polynomial long division, we will try to see if we can reduce the fraction. Indeed, we find

$$f'(-1)=\lim_{x\to -1}\frac{x^3-3x-2}{x+1}=\lim_{x\to -1}(x^2-x-2)=0$$

That is, we have again found that $m_t = 0$ is the slope of the line tangent to the point $(-1,2)$ on the function $f(x) = x^3 - 3x$.

Problem 6:

Find the equation of the line tangent to the point $(-1,2)$ on the function $f(x) = x^3 - 3x$. See Problem 5 above!

Remember from algebra that we can use the *point-slope equation* $y - y_1 = m(x - x_1)$ to find the equation of a line when given one point and the slope.

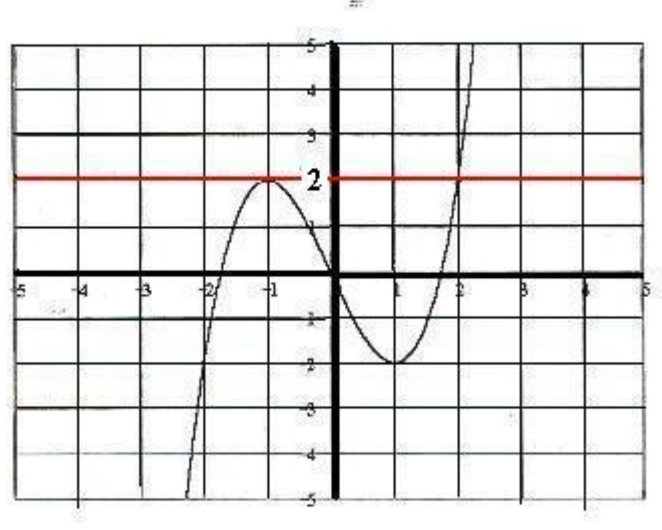
In our case, m is actually $f'(-1)$ from Problem 5 above.

Using the *point-slope equation* of the line, we can now find the equation of the line tangent to the point $(-1,2)$ on the function $f(x) = x^3 - 3x$.

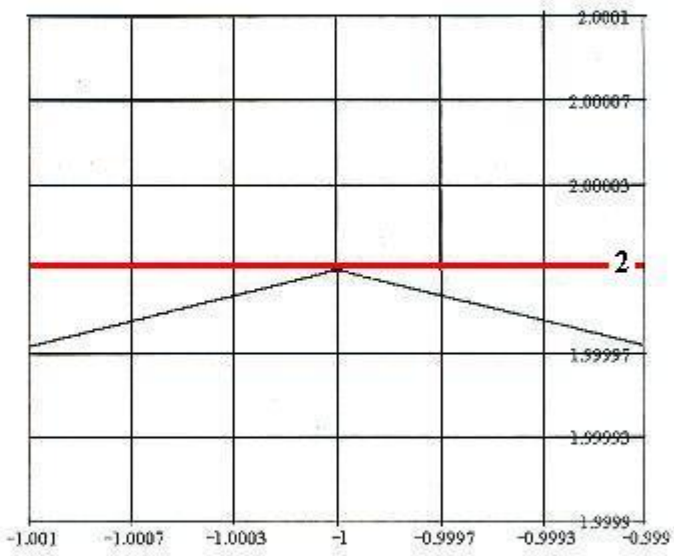
$$y - 2 = 0(x + 1)$$
$$y = 2$$

That is, we have found that $y = 2$ is the equation of the line tangent to the point $(-1,2)$ on the function $f(x) = x^3 - 3x$.

The first picture below shows the graph of $f(x) = x^3 - 3x$ and the graph of the line $y = 2$.



The second picture zooms in on the point $(-1,2)$ and shows that the line $y = 2$ indeed only touches the graph of f only at that point.



2.0001
2.00007
2.00003
2
1.99997
1.99993
1.9999
-1.001
-1.0007
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-1
-0.9997
-0.9993
-0.999