

$$\lim_{x \rightarrow \infty} \int_2^3 \frac{1}{dx} dy$$

THE DEFINITE INTEGRAL

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Please Send Questions and Comments to ingrid.stewart@csn.edu. Thank you!

In this unit we will develop a formula for the area of a **Plane Region** and a definition for the **Definite Integral**. We will also state the **Fundamental Theorem of Calculus**. However, we will begin this unit by introducing a concise notation for sums, which is needed in the development of these concepts. This notation is called **sigma notation** because it uses the upper case Greek letter sigma, written as Σ .

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \dots, a_n$ is written as

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n$$

where k is the **index of summation** and the **upper and lower bounds of summation** are n and 1 .

NOTE: The index of summation is also often denoted to i or j and consists of any integer as long as no term in the sum becomes imaginary or undefined and the lower bound is less than or equal to the upper bound.

$$\sum_{k=1}^n a_k \text{ is pronounced as } \textit{the summation of } a_k \textit{ for } k = 1 \textit{ to } n !$$

Examples:

$$\sum_{j=1}^3 (2^j + 1) = (2^1 + 1) + (2^2 + 1) + (2^3 + 1) = 3 + 5 + 9 = 17$$

$$\sum_{i=1}^3 6i = 6(1) + 6(2) + 6(3) = 6 + 12 + 18 = 36$$

$$\sum_{i=1}^3 6 = 6 + 6 + 6 = 18$$

$$\sum_{k=0}^3 \frac{1}{k} = \frac{1}{0} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$$

NOTE: This sum is undefined due to the selection of the index !!!! We cannot use $k = 0$.

Properties of Summation

1. Let \mathbf{c} be any real number. Then $\sum_{i=1}^n \mathbf{c}a_i = \mathbf{c} \sum_{i=1}^n a_i$
2. $\sum_{i=1}^n (\mathbf{a}_i \pm \mathbf{b}_i) = \sum_{i=1}^n \mathbf{a}_i \pm \sum_{i=1}^n \mathbf{b}_i$

Some Useful Summation Formulas

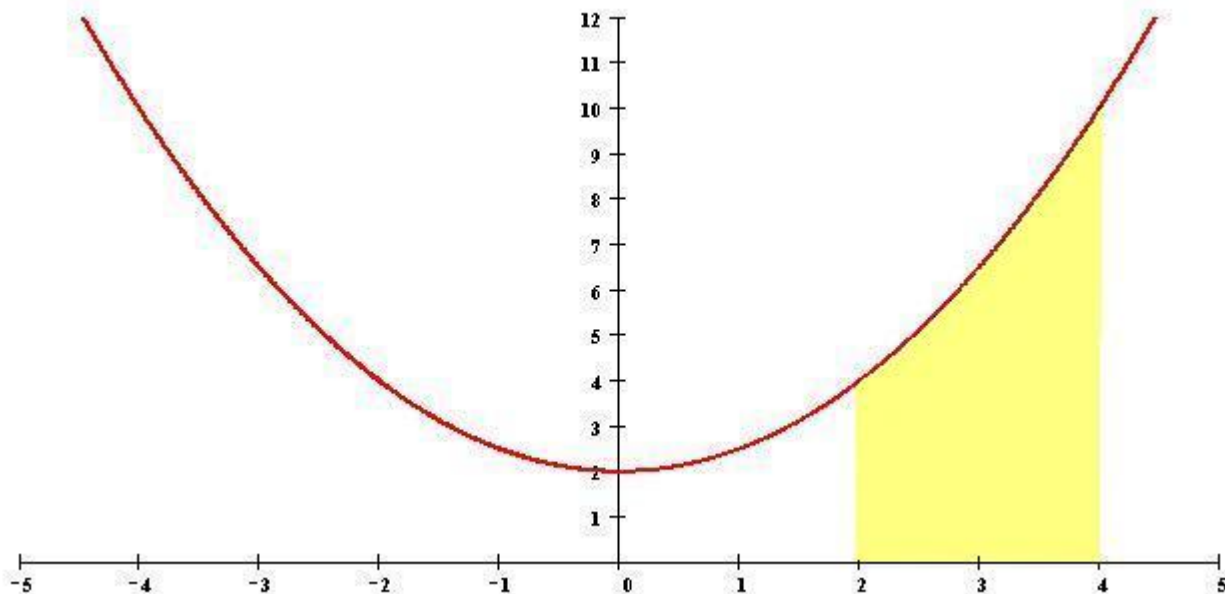
1. Let \mathbf{c} be any real number. Then $\sum_{i=1}^n \mathbf{c} = \mathbf{c}n$
2. $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$



THE AREA OF A PLANE REGION

We mean by the area of a *Plane Region*, an area bounded by the graph of a function f , the x-axis, and some vertical lines $x = a$ and $x = b$, where $a < b$.

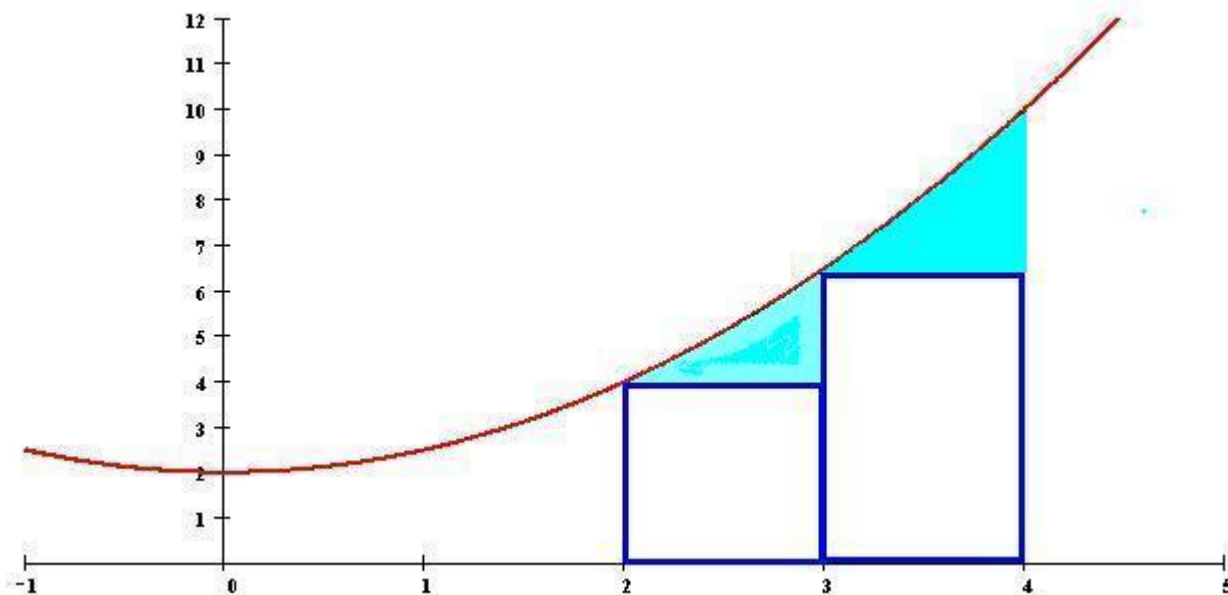
See picture below where $f(x) = \frac{1}{2}x^2 + 2$ and $a = 2$ and $b = 4$.



Since for the area of a *Plane Region* most often no formula exists, we might find an approximation of this area by using rectangles and then adding up them up. If we indeed decide to do this, there would be two ways in which to insert rectangles into the *Plane Region*!

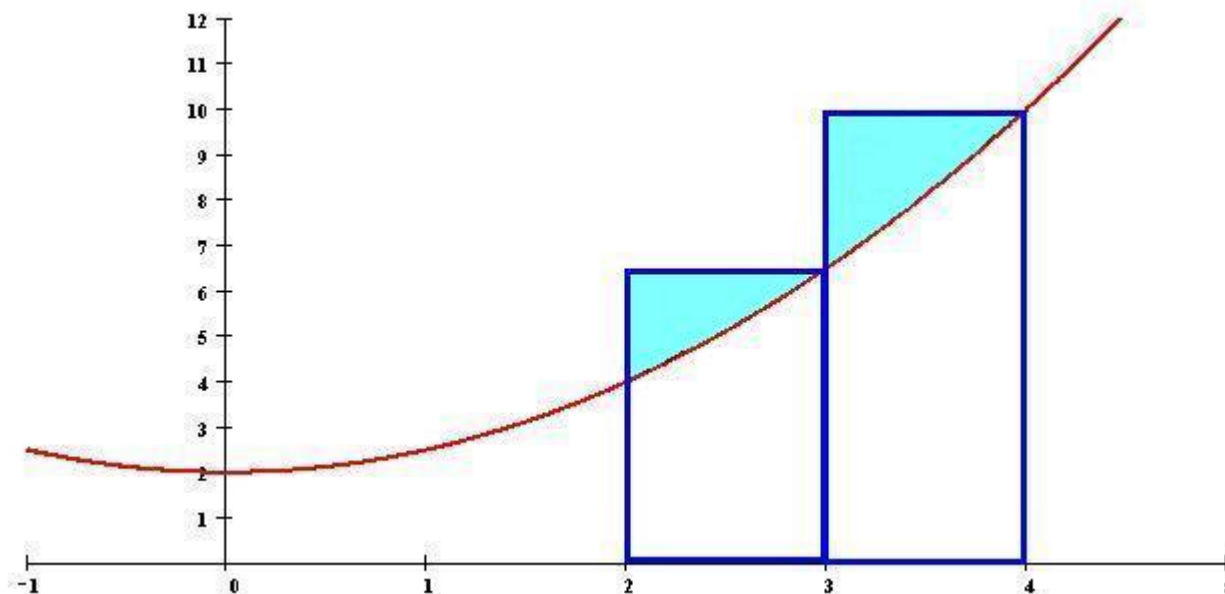
The following picture shows two **inscribed rectangles** of equal length using $f(x) = \frac{1}{2}x^2 + 2$ and $a = 2$ and $b = 4$. The sum of the areas of the inscribed rectangles is called **lower sum** and denoted by (lower case) S_n .

NOTE: The shaded regions show the "error" in the area calculation when using the sum of inscribed rectangles.



The next picture shows two **circumscribed rectangles** of equal length again using $f(x) = \frac{1}{2}x^2 + 2$ and $a = 2$ and $b = 4$. The sum of the areas of the circumscribed rectangles is called **upper sum** and denoted by (upper case) S_n .

NOTE: The shaded regions show the "error" in the area calculation when using the sum of circumscribed rectangles.



We will develop the formula for the area a Plane Region using the above example. If you are interested in finding out how to find the area of a general inscribed and circumscribed rectangle, click on the document entitled "Formulas for the Dimensions of Inscribed and Circumscribed Rectangles Given any Function", which can be found in the online lecture notes under the heading "Unit 16."

To get back to our example, let's go ahead and now find the approximate area of the region between the function $f(x) = \frac{1}{2}x^2 + 2$ and the x-axis on the interval $[2,4]$

(a) using the lower sum of two *inscribed* rectangles

Length:

Please note that we are assuming that each rectangle is of equal length!

It is customary to use Δx (delta x) to denote the length!

$$\Delta x = \frac{4 - 2}{2} = 1$$

Height:

Rather than take formulas, we can easily see from the picture, that the height of the smaller (first) rectangle is $f(2)$ and the height of the larger (second) rectangle is $f(3)$.

Lower Sum:

$$s_n = 1 \cdot f(2) + 1 \cdot f(3) = 4 + 6.5 = 10.5$$

(b) using the upper sum of two *circumscribed* rectangles

Length:

$$\Delta x = \frac{4 - 2}{2} = 1$$

Height:

As we can see from the picture, the height of the smaller (first) rectangle is $f(3)$ and the height of the larger (second) rectangle is $f(4)$.

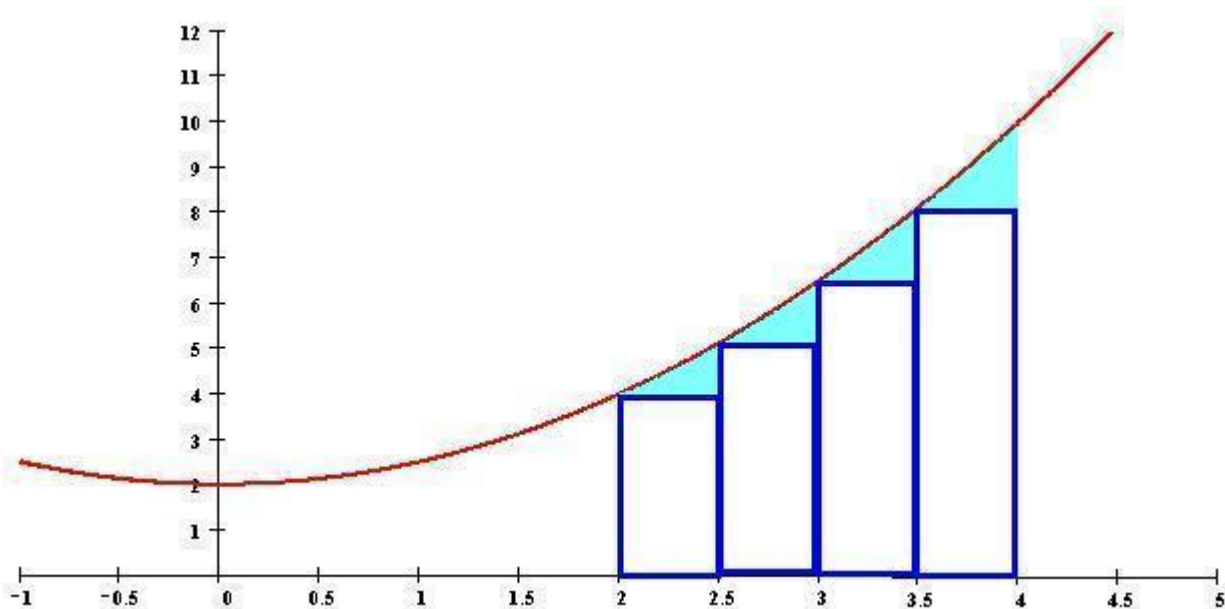
Upper Sum:

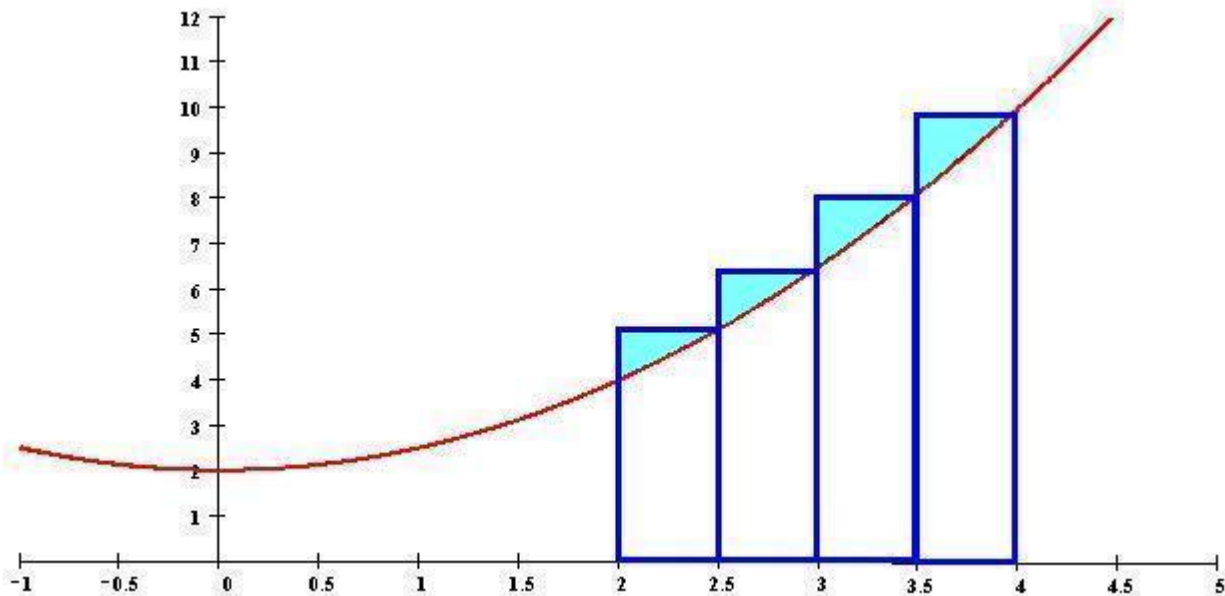
$$S_n = 1 \cdot f(3) + 1 \cdot f(4) = 6.5 + 10 = 16.5$$

Using only two inscribed and two circumscribed rectangles seems to result in a large "error" in the actual area calculation.

How about if we find the approximate area of the region between the function

$f(x) = \frac{1}{2}x^2 + 2$ and the x-axis on the interval $[2, 4]$ by using four rectangles? Look at the following two pictures!





(a) using the lower sum of four *inscribed* rectangles

Length:

$$\Delta x = \frac{4 - 2}{4} = \frac{1}{2} = 0.5$$

Height:

As we can see from the picture that the heights of the rectangles in order are $f(2)$, $f(2.5)$, $f(3)$, and $f(3.5)$.

Lower Sum:

$$\begin{aligned} s_n &= 0.5 \cdot f(2) + 0.5 \cdot f(2.5) + 0.5 \cdot f(3) + 0.5 \cdot f(3.5) \\ &= 0.5(4) + 0.5(5.125) + 0.5(6.5) + 0.5(8.125) \end{aligned}$$

and $s_n = 2 + 2.562 + 3.25 + 4.062 = 11.874$

(b) using the upper sum of four *circumscribed* rectangles

Length:

$$\Delta x = \frac{4 - 2}{4} = \frac{1}{2} = 0.5$$

Height:

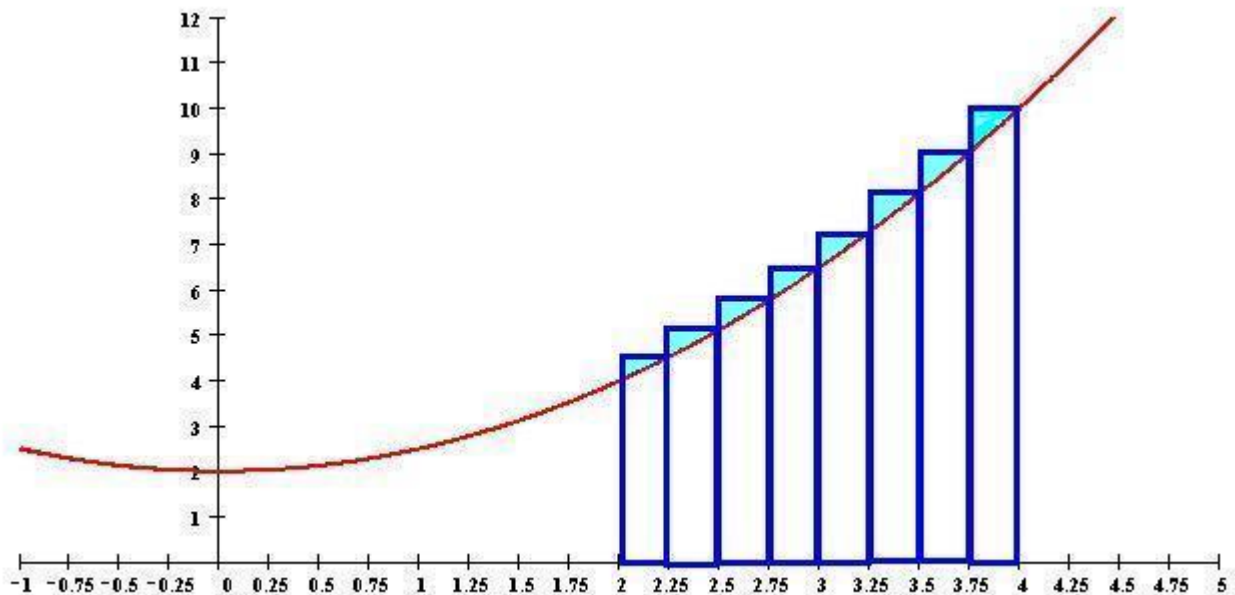
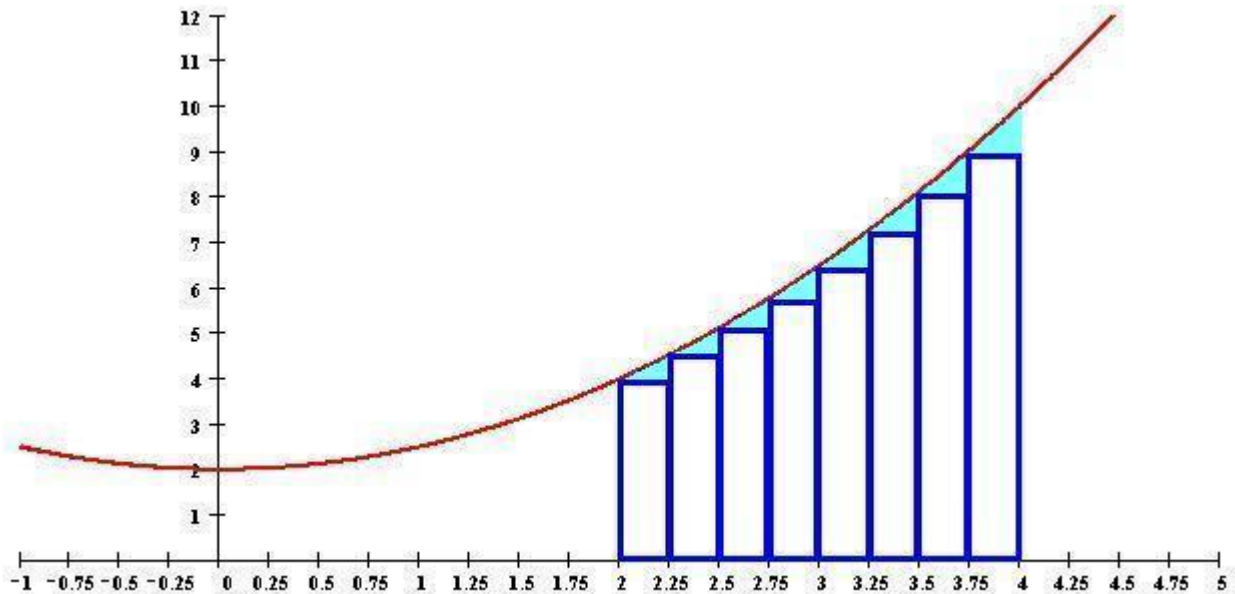
As we can see from the picture that the heights of the rectangles in order are $f(2.5)$, $f(3)$, $f(3.5)$, and $f(4)$.

Upper Sum:

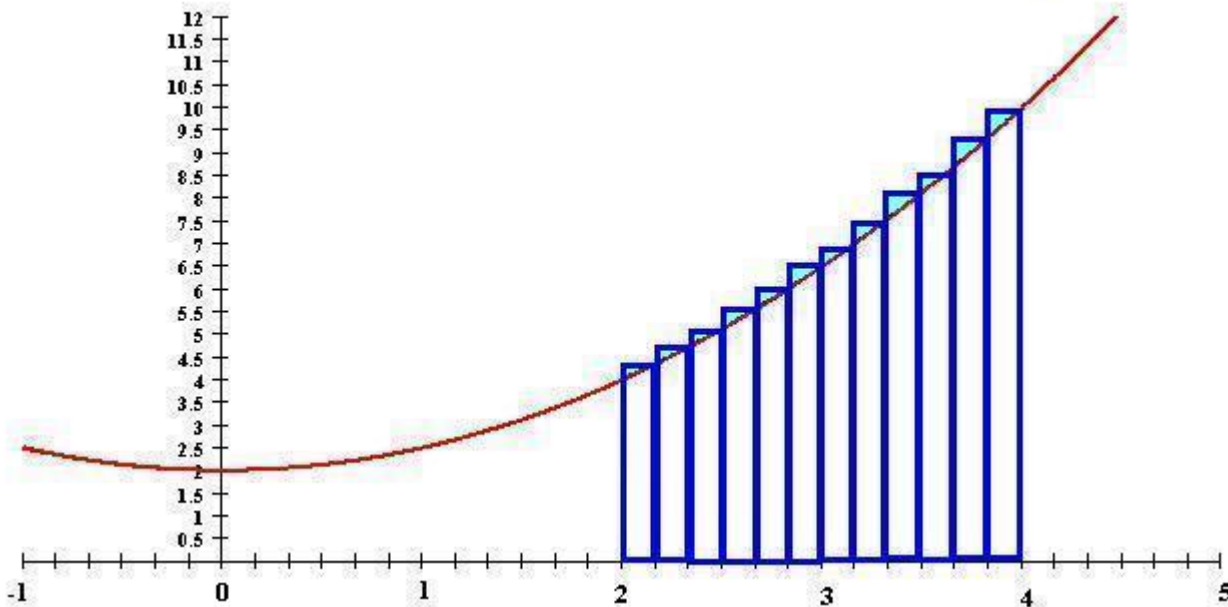
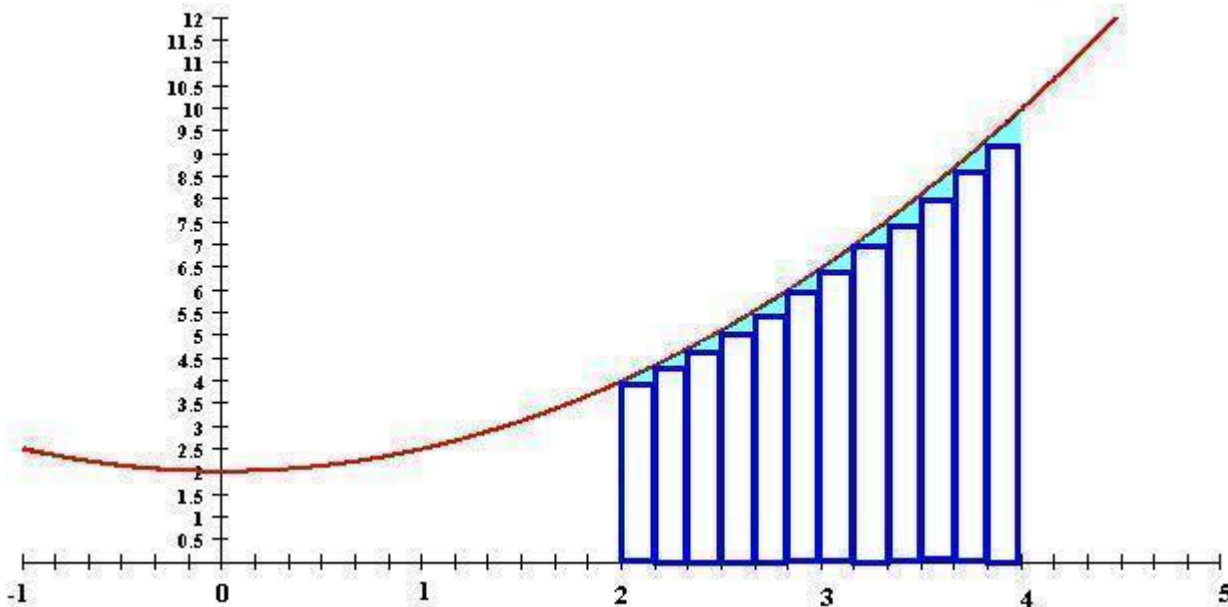
$$\begin{aligned} S_n &= 0.5 \cdot f(2.5) + 0.5 \cdot f(3) + 0.5 \cdot f(3.5) + 0.5 \cdot f(4) \\ &= 0.5(5.125) + 0.5(6.5) + 0.5(8.125) + 0.5(10) \end{aligned}$$

$$S_n = 2.562 + 3.25 + 4.062 + 4 = 13.874$$

Let's look at some more pictures. The next two pictures show the region between the function $f(x) = \frac{1}{2}x^2 + 2$ and the x-axis on the interval $[2, 4]$ partitioned into eight subintervals providing for eight rectangles.



The following two pictures show the region between the function $f(x) = \frac{1}{2}x^2 + 2$ and the x-axis on the interval $[2,4]$ partitioned into twelve subintervals providing for twelve rectangles.



Looking at the calculations and pictures above should give us an inkling that the "error" in the area calculation using rectangles might get smaller and smaller the more rectangles we are trying to "squeeze" into the region.

As a matter of fact, couldn't we say intuitively that if we were to "squeeze" infinitely many rectangles (inscribed or circumscribed) into the region, the "error" in the area calculation might "vanish" and we will have found the actual area ???

Let's try it and see what happens !!!!!

1. We must find the approximate area of the region between the function $f(x) = \frac{1}{2}x^2 + 2$ and the x-axis on the interval $[2,4]$ using n rectangles.

That is,

$$s_n = 13\frac{1}{3} - \frac{6}{n} + \frac{2}{3n^2} \text{ (lower sum)}$$

and
$$S_n = 13\frac{1}{3} + \frac{6}{n} + \frac{2}{3n^2} \text{ (upper sum)}$$

If you are interested in finding out how to find these two sums, click on the document entitled "Finding the Upper and Lower Sum of "n" Rectangles Given a Quadratic Function", which can be found in the online lecture notes under the heading "Unit 16."

2. Next, we will find the limit of the upper and lower sum as n approaches infinity.

That is,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(13\frac{1}{3} - \frac{6}{n} + \frac{2}{3n^2} \right) = 13\frac{1}{3}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(13\frac{1}{3} + \frac{6}{n} + \frac{2}{3n^2} \right) = 13\frac{1}{3}$$

Observe that as infinitely many rectangles get squeezed into the region between the function $f(x) = \frac{1}{2}x^2 + 2$ and the x-axis on the interval $[2,4]$ the lower sum equals the upper sum!

The resulting limit is considered to be the EXACT area of the region! That is, the area of our region is exactly $13\frac{1}{3}$ square units in size.

Hopefully, after the lengthy discussion above you are now better able to understand the formula of the area of a *Plane Region* between the graph of **ANY** function and the x-axis on any given interval $[a, b]$.

Formula for the Area of a Plane Region

Let f be a continuous **nonnegative** function defined on the interval $[a, b]$.

Then, the area of the region bounded by the graph of f , the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \overbrace{f(c_i)}^{\text{height}} \overbrace{\Delta x}^{\text{length}}, \text{ where } \Delta x = \frac{b-a}{n} \text{ and } c_i \text{ any value in each subinterval.}$$

NOTES:

1. Sigma indicates that we are summing the areas of "n" rectangles.
2. To find the height, we are using any value c_i contained within each subinterval defining a rectangle. From our work with the quadratic function example, we found that the upper sum equals the lower sum when $n \rightarrow \infty$. Therefore, we can conclude that it does not matter what x-value we used for the height calculation when the number of rectangles approaches infinity.



THE DEFINITION OF THE DEFINITE INTEGRAL

The definition of the Definite Integral will be developed by using the formula for the area of a *Plane Region*

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \overbrace{f(c_i)}^{\text{height}} \overbrace{\Delta x}^{\text{length}}$$

- First we will observe that if $n \rightarrow \infty$, then the length Δx of each rectangle approaches 0 . Therefore, we can write the area formula as

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x$$

- Next, let's assume that the lengths of the rectangles we are "squeezing" into a *Plane Region* are NOT equal.

That is, we want to partition the interval $[a, b]$ into n subintervals of varying lengths, such as $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$, where $x_0 = a$ and $x_n = b$.

As a result, we denote the length of the first rectangle by

$$\Delta x_1 = x_1 - x_0,$$

the length of the second rectangle by

$$\Delta x_2 = x_2 - x_1,$$

and the length of the i th rectangle by

$$\Delta x_i = x_i - x_{i-1}, \text{ where } i = 1, 2, 3, \dots, n.$$

The length of the longest rectangle is called the **norm** of the partition, which we will denote $\|P\|$.

Therefore, we can write the area formula as $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$

This limit plays such a major role in mathematics that a special name and symbol is given to it. It is called the **Definite Integral** and is denoted by

$$\int_a^b f(x) dx$$

pronounced as "the integral of $f(x)$ with respect to x from a to b ."

- The symbol \int is an elongated letter S and stands for "Sum".
- a is called the **lower limit of integration** and the number b is called the **upper limit of integration**.

NOTE: The process of evaluating the *definite integral* is called **integration** and $f(x)$ is called the **integrand**.

Definition of the Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

exists, then the **Definite Integral** of the function f is defined as

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

Uses of the Definite Integral

1. The *definite integral* can be used to find the area bounded by the graph of a function and the x-axis.

That is, we can now denote the area between a function and the x-axis in the interval $[a, b]$ as

$$A = \int_a^b f(x) dx$$

given that the y-values are **nonnegative** for every x in the interval

2. Since every *definite integral* is a particular limiting value, it is used frequently in the sciences, in business, and in engineering. It allows us to find quantities such as the center of masses, fluid pressure and force, price elasticity of demand (measures the responsiveness of a quantity demanded to a change in price), etc.

NOTE: Unless you are calculating the area between a function and the x-axis on some interval, you do NOT have to worry about encountering negative y-values.



THE FUNDAMENTAL THEOREM OF CALCULUS

Obviously, it is impossible to find

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

given that all rectangles are of different lengths.

Fortunately, the mathematicians Isaac Newton and Gottfried Leibniz independently discovered a close connection between *derivatives* and the *definite* integral, which makes the calculations of the Definite Integral quite manageable.

NOTE: Actually we will see that differentiation and integration are inverse processes!

This connection is stated in a theorem called the **Fundamental Theorem of Calculus**. The proof of this theorem is shown in the document entitled "Proof of the Fundamental Theorem of Calculus", which can be found in the online lecture notes under the heading "Unit 16."

The Fundamental Theorem of Calculus

If a function f is continuous on the interval $[a, b]$ and F is an **antiderivative** of f on the interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Definition of Antiderivative

A function F is an **antiderivative** of a function f , if $F'(x) = f(x)$.

For example:

Given $F(x) = x^2 + C$, where C is any real number and $f(x) = 2x$.

Since $F'(x) = 2x = f(x)$, we can say that $x^2 + C$ is an antiderivative of $2x$.

NOTE: We say that $F(x) = x^2 + C$ is a *family of antiderivatives* of the function $f(x) = 2x$.

Before we can use the Fundamental Theorem of Calculus to evaluate definite integrals, we must first spend the next four units practicing how to find antiderivatives.